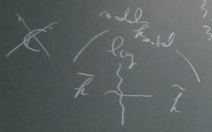
bi-anab. vs mono-anab.

cf. [AbsTopIII] Rem 3.7.3
3.7.4
3.7.5



§9. Hodge theaters – Arithmetic Upper Half Plane

§9.0 Some Conventions

Def 9.1 $\mathcal{C}$: cat. $A, B \in \mathrm{Ob}(\mathcal{C})$

(1). – a **poly-morphism** $A \to B$
$\overset{\text{def}}{\Leftrightarrow}$ a subset of $\mathrm{Hom}_{\mathcal{C}}(A,B)$

– the **full poly-morphism** $A \to B$
$\overset{\text{def}}{\Leftrightarrow}$ the poly-morph given by $\mathrm{Hom}_{\mathcal{C}}(A,B)$

– poly-isom ← a subset of $\mathrm{Isom}_{\mathcal{C}}(A,B)$

– full poly-isom ← given by $\mathrm{Isom}_{\mathcal{C}}(A,B)$

(2). a **capsule** of obj's of $\mathcal{C}$
$\overset{\text{def}}{\Leftrightarrow}$ a finite collection of $\mathrm{Obj}(\mathcal{C})$
$\{A_j\}_{j \in J}$ $J := \pi_0(\{A_j\}_{j \in J})$
$|J|$-capsule

$\rightsquigarrow$ cat. $\mathrm{Capsule}(\mathcal{C})$

a **morph. of capsules** $\{A_j\}_{j \in J} \to \{A'_{j'}\}_{j' \in J'}$
$\overset{\text{def}}{\Leftrightarrow} \begin{cases} \cdot\ \iota : J \hookrightarrow J' \\ \cdot\ A_j \to A'_{\iota(j)} \in \mathrm{Hom}_{\mathcal{C}} \text{ for each } j \in J \end{cases}$

a **capsule-full poly-morph**

$$\{A_j\}_{j \in J} \longrightarrow \{A'_{j'}\}_{j' \in J'}$$

$\mathrm{Ob}(\mathrm{Capsule}(\mathcal{C}))$

fixed
$\downarrow$
$\iota : J \hookrightarrow J'$

: $\Rrightarrow$: the poly-morph assoc. to some

: $\Rrightarrow$: **capsule-full poly-isom**

: $\Rrightarrow$: capsule-full poly-morph
s.t. $\begin{cases} \iota : \text{bij} \\ A_j \xrightarrow{\sim} A'_{\iota(j)} \\ \text{some isom through} \end{cases}$

:

$\mathcal{B}^{\text{temp}}(\Pi)$

$\mathcal{C} \longrightarrow \mathcal{C}'$
morph
$\Updownarrow$ def
an isom. class of functors
$\longleftarrow$

$\mathcal{C}^0 \longrightarrow (\mathcal{C}')^0$
$\Updownarrow$ def
$(\mathcal{C}^0)^\top \longrightarrow ((\mathcal{C}')^0)^\top$

the cat. of formal countable coproducts

$\perp \rightsquigarrow \top$

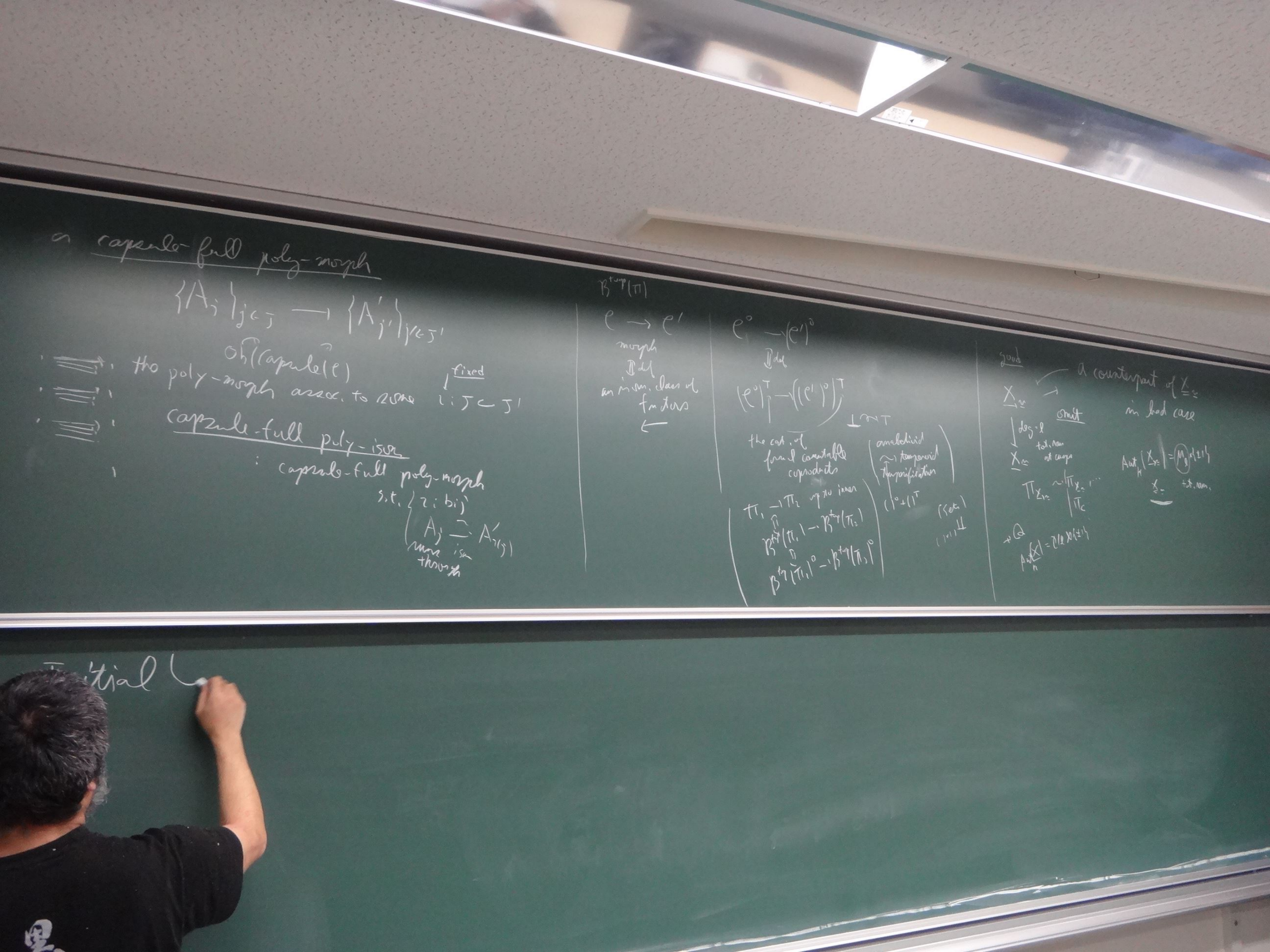
(anabelioid $\rightsquigarrow$ temperoid
temporification
$(\ )^0 + (\ )^\top$)

($\Pi_1 \to \Pi_2$ up to inner
$\Rightarrow$
$\mathcal{B}^{\text{tmp}}(\Pi_1) \to \mathcal{B}^{\text{tmp}}(\Pi_2)$
$\Rightarrow$
$\mathcal{B}^{\text{tmp}}(\Pi_1)^0 \to \mathcal{B}^{\text{tmp}}(\Pi_2)^0$)

good

$$\begin{tikzcd} \underline{X}_{\underline{v}} \arrow[d, "\deg\text{-}l"] \\ \underline{X}_{\underline{v}} \end{tikzcd}$$

tot. ram at cusps



a capsule-full poly-morph

$\{A_j\}_{j\in J} \longrightarrow \{A'_{j'}\}_{j'\in J'}$

$\mathrm{Ob}(\mathrm{capsule}(\mathcal{C}))$

$\Rightarrow$: the poly-morph assoc. to some $\iota : J \hookrightarrow J'$ fixed

capsule-full poly-isom

: capsule-full poly-morph s.t. $\{\iota : \mathrm{bij}\}$, $A_j \xrightarrow{\sim} A'_{\iota(j)}$ isom through

$\mathcal{B}^{temp}(\Pi)$

$\mathcal{C} \longrightarrow \mathcal{C}'$ morph $\Updownarrow$ def an isom. class of functors $\longleftarrow$

$\mathcal{C}^0 \longrightarrow (\mathcal{C}')^0$ $\Updownarrow$ def $(\mathcal{C}^0)^\top \longrightarrow ((\mathcal{C}')^0)^\top$

$\perp \rightsquigarrow \top$

the cat. of formal countable coproducts

(anabelioid $\rightsquigarrow$ temperoid, temporification)

$(\;)^0 + (\;)^\top$

(Sets) $(\;)^{\amalg}$

$\Pi_1 \to \Pi_2$ up to inner

$\mathcal{B}^{temp}(\Pi_1) \to \mathcal{B}^{temp}(\Pi_2)$

$\mathcal{B}^{temp}(\Pi_1)^0 \to \mathcal{B}^{temp}(\Pi_2)^0$

good

$\underline{X}_v$ — a counterpart of $\underline{\underline{X}}_v$ in bad case — omit

$\downarrow$ deg $= \ell$, tot. ram. at cusps

$\underline{X}_v$

$\Pi_{\underline{X}_v} \rightsquigarrow \Pi_{\underline{X}_v}, \ldots$ $\mid \Pi_c$

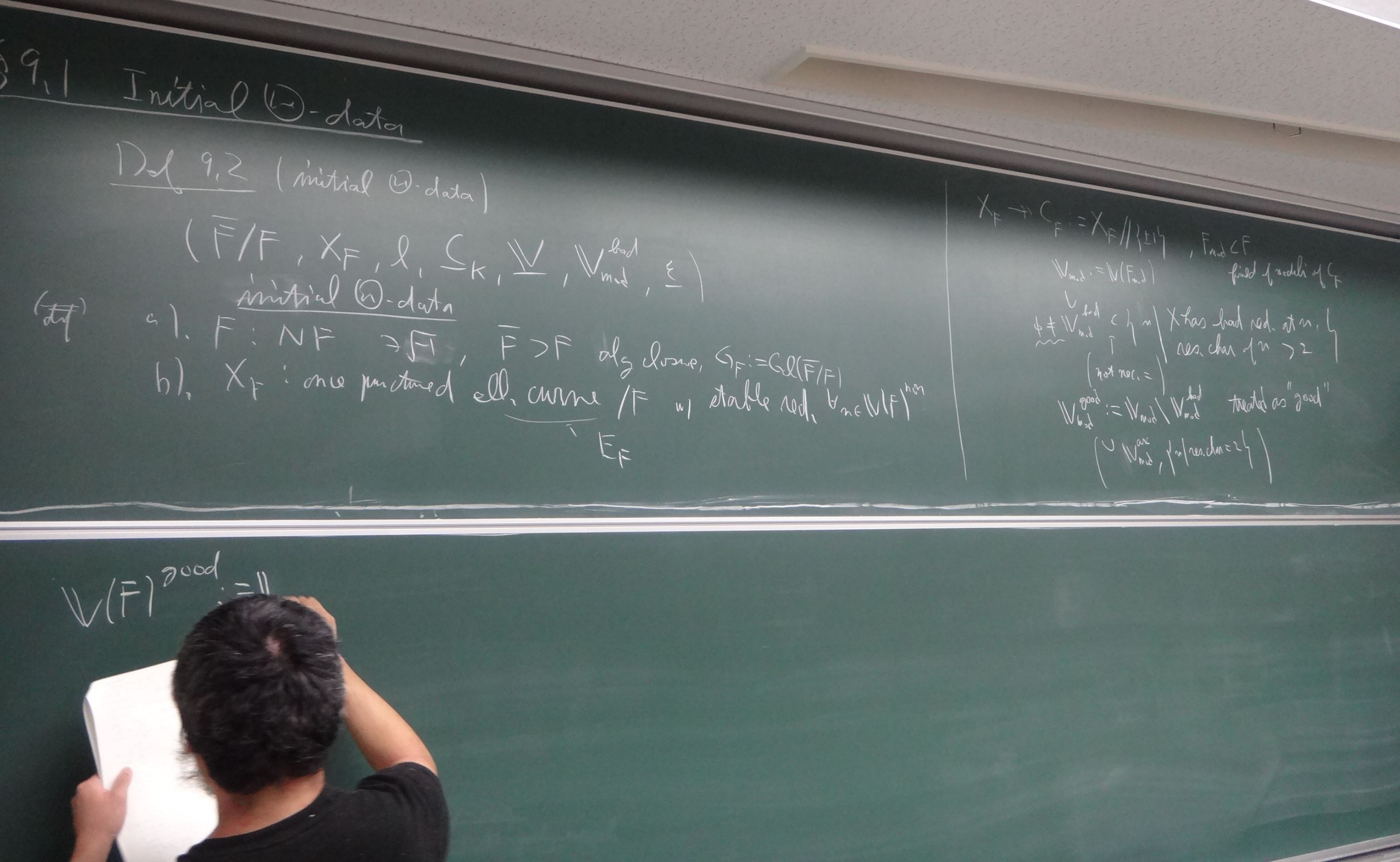
$\mathrm{Aut}_k(\underline{X}_v) = \boxed{\mu_\ell} \rtimes \{\pm 1\}$

$\underline{\underline{X}}_v$ tot. ram.

$\not\ni \mathbb{Q}$

$\mathrm{Aut}_k(\underline{X}) = \mathbb{Z}/\ell\mathbb{Z} \rtimes \{\pm 1\}$

Initial Θ



§9.1 Initial Θ-data

Def 9.2 (initial Θ-data)

$$(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \mathbb{V}_{mod}^{bad},\ \underline{\epsilon})$$

$\underbrace{}$ initial Θ-data

(def) a). F : NF $\ni \sqrt{-1}$, $\overline{F} \supset F$ alg closure, $G_F := \mathrm{Gal}(\overline{F}/F)$

b). X_F : once punctured $\underbrace{\text{ell. curve}}_{E_F}$ /F w/ stable red. $\forall_{v \in \mathbb{V}(F)^{non}}$

$X_F \twoheadrightarrow C_F := X_F /\!/ \{\pm 1\}$, $F_{mod} \subset F$ field of moduli of C_F

$\mathbb{V}_{mod} := \mathbb{V}(F_{mod})$

$\cup$

$\underset{\sim\sim}{\emptyset \neq} \mathbb{V}_{mod}^{bad} \underset{\uparrow}{\subset} \{ v \mid X \text{ has bad red. at } v,\ \text{res. char of } v > 2 \}$

(not nec. =)

$\mathbb{V}_{mod}^{good} := \mathbb{V}_{mod} \setminus \mathbb{V}_{mod}^{bad}$ treated as "good"

$(\cup\ \mathbb{V}_{mod}^{arc},\ \{ v \mid \text{res. char} = 2 \})$

$\mathbb{V}(F)^{good} := $

$\mathbb{V}(F)^{good} := \mathbb{V}_{mod}^{good} \times \mathbb{V}(F)$

$\mathbb{V}(F)^{bad} := \mathbb{V}_{mod}^{bad} \times_{\mathbb{V}_{mod}} \mathbb{V}(F)$

- F/F_{mod} Galois
- ∀ pts of $E_F[2 \cdot 3]$: rational /F $\left(\supseteq \rightarrow K/F_{mod} \text{ Galois} \right)$

c) $l \geq 5$ prime $\quad G_F \longrightarrow GL_2(\mathbb{F}_l)$

Image ↑ contains $SL_2(\mathbb{F}_l)$

$K := F(E_F[l])$

outer hom

(↖ up to isom)

- l is prime to | $\forall v \in \mathbb{V}_{mod}^{bad}$

| ord_{F_v} of q-parameter at $\mathbb{V}(F)^{bad}$

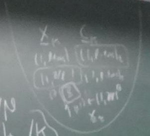
ord([illegible] of F_v) = 1

d). $\underline{C}_K$: hyperb. orbicurve of type $(1, l\text{-tors})_{\pm}$ /K

w/ K-core $C_K := C_F \times_F K$

$\left(c) \Rightarrow \left(C_F \rightsquigarrow \underline{C}_K \text{ up to isom} \right) \right)$ $\quad ((1, \mathbb{Z}/l\mathbb{Z})\text{-type})$

$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} / \pm \qquad$ c) Image $\supset SL_2(\mathbb{F}_l)$



$\rightarrow \underline{X}_K$ up to isom /K, type $(1, l\text{-tors})$

$$\begin{tikzcd} \underline{X}_K \arrow[r] \arrow[d, "\pm 1"'] & \underline{X}_F \arrow[d, "\pm 1"] \\ \underline{C}_K \arrow[r] & \underline{C}_F \end{tikzcd}$$

$\Delta_C \supset \Delta_{\underline{C}} \supset \Delta_{\underline{X}}$

$\underline{X}_F$, $\underline{C}_F$ of type $(1, l\text{-tors})_{\pm}$

Π

e).

e). $\underline{\mathbb{V}} \subset \mathbb{V}(K)$ subset

s.t.
$$\begin{tikzcd}
\mathbb{V}(K) \arrow[d] & \supset \underline{\mathbb{V}} \arrow[dl, "\wr"] & \ni \underline{v} \arrow[d, dash] \\
\mathbb{V}_{\mathrm{mod}} & & \ni v
\end{tikzcd}$$

$\underline{\mathbb{V}}^{\mathrm{non}} := \underline{\mathbb{V}} \cap \mathbb{V}(K)^{\mathrm{non}}$

arc

'good'

'bad'

$\bar{\underline{v}} \in \mathbb{V}(\bar{F}) / \underline{v}$

$$\begin{tikzcd}
\underline{\underline{X}}_{\bar{\underline{v}}} \arrow[r, "l\text{-sp.cov}", "\Delta_\Theta"'] \arrow[d, "\pm 1"] & \underline{X}_{\underline{v}} \arrow[r, "\circlearrowleft \cong \mathbb{Z}/l"] \arrow[d, "\pm 1"] & X_{\underline{v}} \arrow[d, "\pm 1"] \\
(\underline{\underline{C}}_{\bar{\underline{v}}}) \arrow[r] & \underline{C}_{\underline{v}} \arrow[r] & C_{\underline{v}}
\end{tikzcd}$$

$^{1:2}\,\mathbb{Z}/l(1)$

$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$

$\Downarrow$

$\underline{C}_{\underline{v}}$: of $(1, \mathbb{Z}/l)_{\pm}$-type

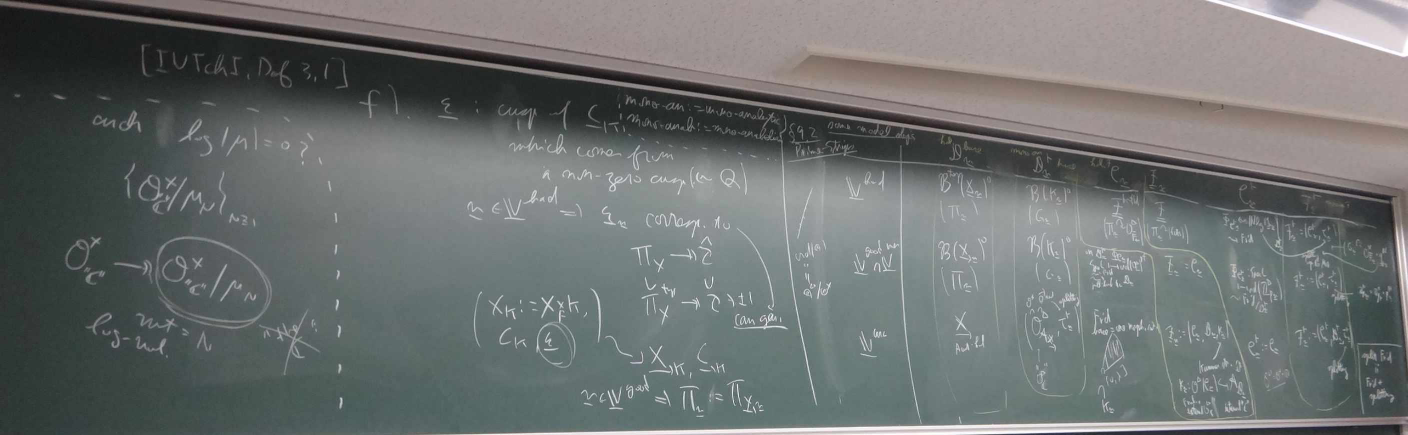
$\left(\text{b)} \rightsquigarrow \overset{\text{upto } \lambda}{\bar{E}(2)} \text{ : rat} \right.$
$\left. \rightsquigarrow K_{\underline{v}} = \ddot{K}_{\underline{v}} \right)$

$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$

[EtTh, §2]

$\underline{\underline{X}}_{\bar{\underline{v}}}$ has a model $\underline{\underline{\mathfrak{X}}}_{\underline{v}} / K_{\underline{v}}$ of type $(1, (\mathbb{Z}/l)^{\Theta})$

$\Pi_{\underline{v}} := \Pi^{\mathrm{tmp}}_{\underline{\underline{\mathfrak{X}}}_{\underline{v}}}$



[IUTchI, Def 3.1]

and $\log|\mu| = 0$?

$\{\mathcal{O}^{\times}_{\text{"}\mathbb{C}\text{"}}/\mu_N\}_{N\geq 1}$

$\mathcal{O}^{\times}_{\text{"}\mathbb{C}\text{"}} \twoheadrightarrow \mathcal{O}^{\times}_{\text{"}\mathbb{C}\text{"}}/\mu_N$

log-mul. $\text{mult}^{\times} = \Lambda$

f). $\underline{\epsilon}$: cusp of $\underline{C}_{K}$ which comes from a non-zero cusp (in Q)

mono-an := mono-analytic

mono-anab := mono-anabelian

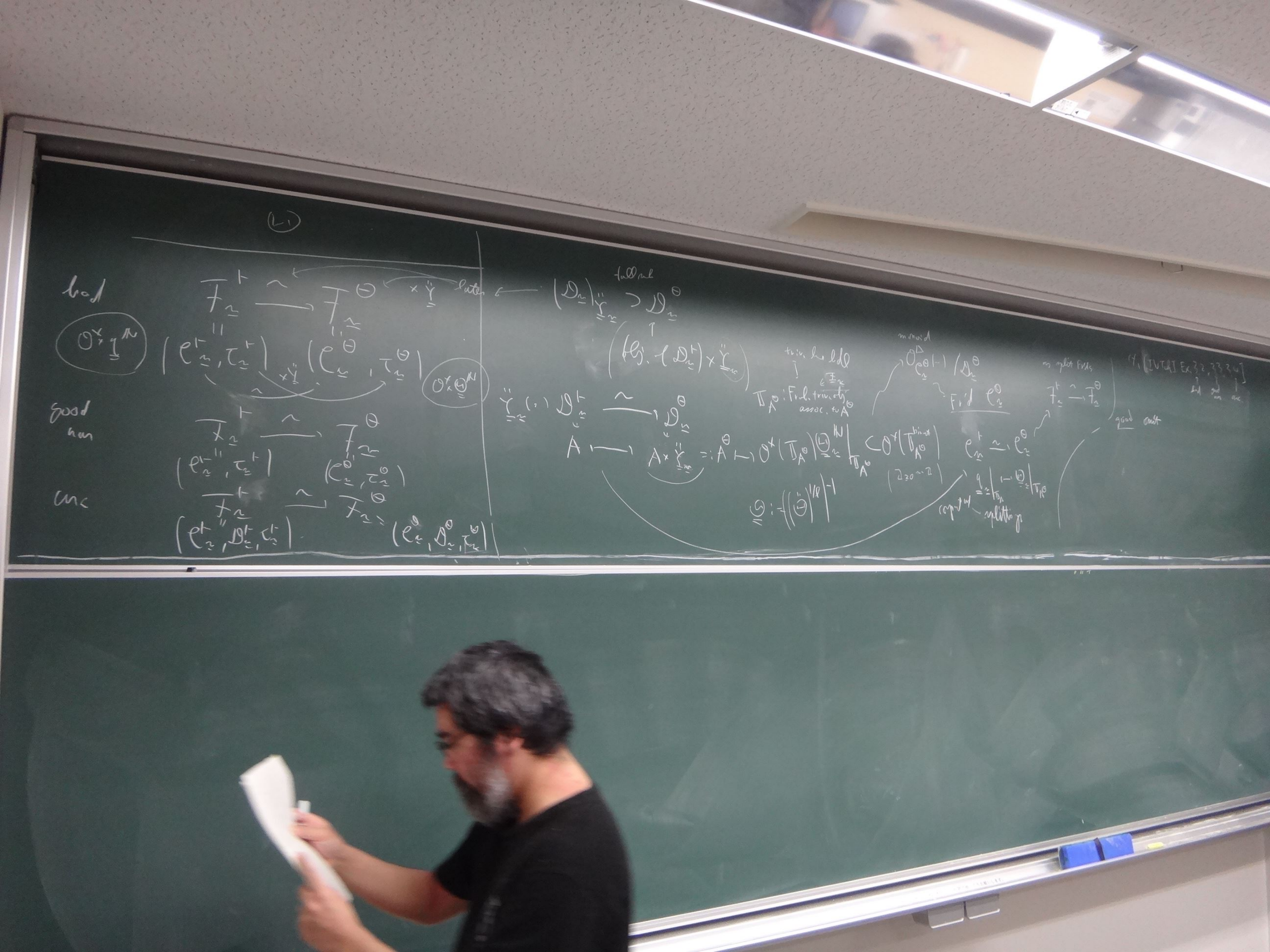
$\underline{v} \in \underline{\mathbb{V}}^{\text{bad}} \Rightarrow \underline{\epsilon}_{\underline{v}}$ corresp. to

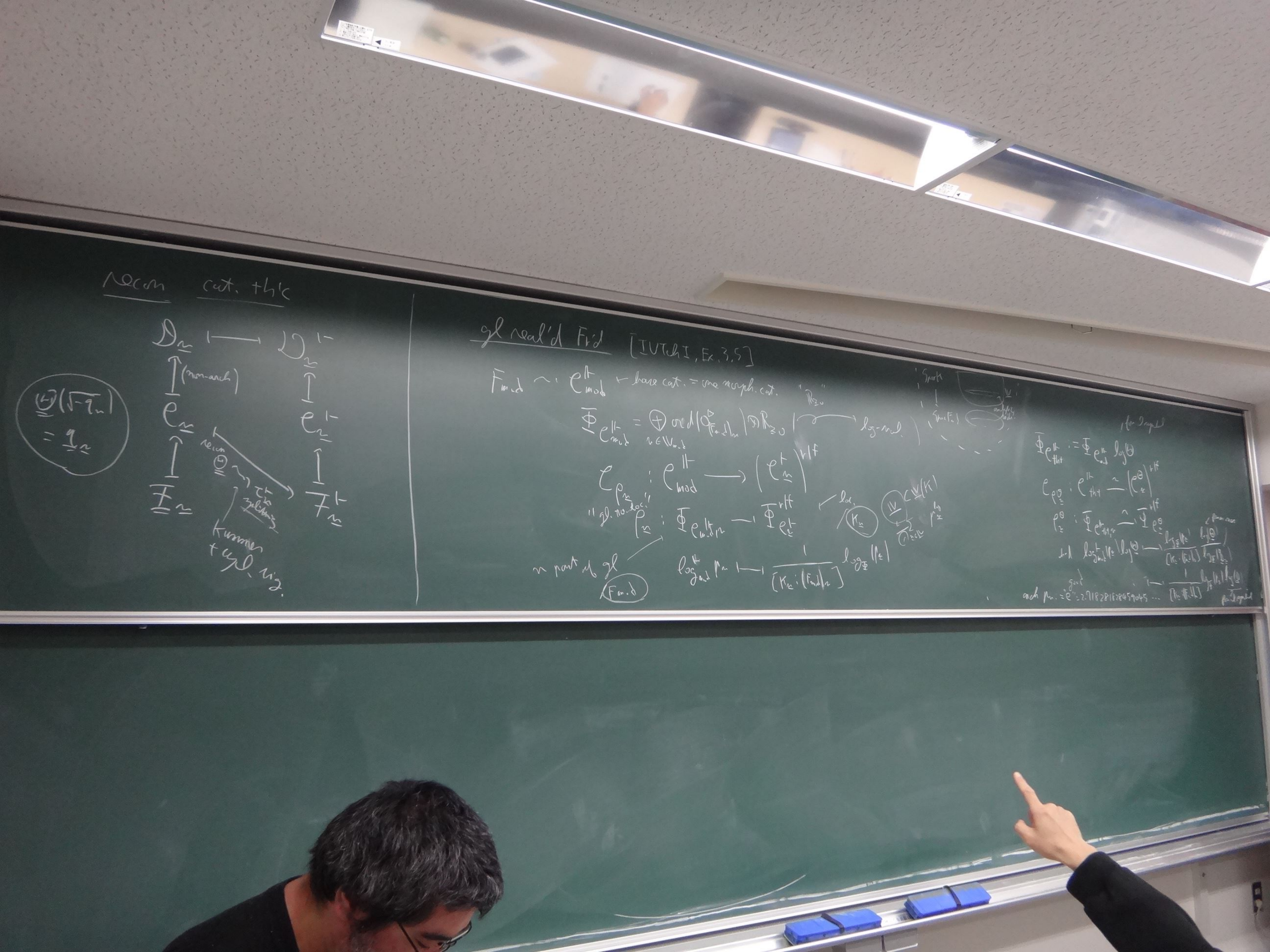
$$\begin{tikzcd}
\Pi_X \arrow[r, two heads] & \hat{\mathbb{Z}} \\
\Pi_X^{\text{tp}} \arrow[u, hook] \arrow[r, two heads] & \mathbb{Z} \arrow[u, phantom, "\cup"] \ni \pm 1
\end{tikzcd}$$

can gen.

$(X_K := X \times_F K,\ C_K,\ \underline{\epsilon}) \rightsquigarrow \underline{X}_K,\ \underline{C}_K$

$\underline{v} \in \underline{\mathbb{V}}^{\text{good}} \Rightarrow \Pi_{\underline{v}} := \Pi_{\underline{X}_{\underline{v}}}$


bad
good non
unc
later
full rel
monoid
[IUTchI] Ex.3.2, 3.3, 3.4
good



recon cat. th'c

$$\underline{\underline{\Theta}}(\sqrt{-q_{\underline{v}}}) = \underline{\underline{q}}_{\underline{v}}$$

$$\begin{tikzcd}
\mathcal{D}_{\underline{v}} \arrow[r, mapsto] & \mathcal{D}^{\vdash}_{\underline{v}} \\
\mathcal{C}_{\underline{v}} \arrow[u, hook, "\text{(non-arch)}"'] \arrow[dr, mapsto, "\text{recon}"] & \mathcal{C}^{\vdash}_{\underline{v}} \arrow[u, hook] \\
\mathcal{F}_{\underline{v}} \arrow[u, hook] & \mathcal{F}^{\vdash}_{\underline{v}} \arrow[u, hook]
\end{tikzcd}$$

$\underline{\underline{\Theta}} \rightsquigarrow \tau^{\vdash}$ splitting

Kummer + cycl. rig.

gl real'd Fr'd [IUTchI, Ex. 3.5]

$F_{mod} \rightsquigarrow \mathcal{C}^{\Vdash}_{mod}$ ← base cat. = one morph. cat. "$\mathbb{R}_{\geq 0}$"

$$\Phi_{\mathcal{C}^{\Vdash}_{mod}} = \bigoplus_{v \in \mathbb{V}_{mod}} \mathrm{ord}(\mathcal{O}^{\triangleright}_{(F_{mod})_v}) \otimes \mathbb{R}_{\geq 0} \quad (\hookrightarrow \text{log-vol.})$$

$$\mathcal{C}_{\rho_{\underline{v}}} : \mathcal{C}^{\Vdash}_{mod} \longrightarrow (\mathcal{C}^{\vdash}_{\underline{v}})^{rlf}$$

"gl. to loc."

$$\rho_{\underline{v}} : \Phi_{\mathcal{C}^{\Vdash}_{mod},v} \longrightarrow \Phi^{rlf}_{\mathcal{C}^{\vdash}_{\underline{v}}}$$

v part of gl (F_{mod})

$$\log^{\Vdash}_{mod} p_v \longmapsto \frac{1}{[K_{\underline{v}} : (F_{mod})_v]} \log_{\Phi}(p_{\underline{v}})$$

loc. ($K_{\underline{v}}$)

$\underline{\mathbb{V}} \subset \mathbb{V}(K)$, $\underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{mod}$, $p^{\log}_{\underline{v}}$

Spec K

Spec F_{mod}

$\underline{\mathbb{V}}$

(for Θ-reg)

$$\Phi_{\mathcal{C}^{\Vdash}_{tht}} := \Phi_{\mathcal{C}^{\Vdash}_{mod}} \log(\Theta)$$

$$\mathcal{C}_{\rho^{\Theta}_{\underline{v}}} : \mathcal{C}^{\Vdash}_{tht} \xrightarrow{\sim} (\mathcal{C}^{\Theta}_{\underline{v}})^{rlf}$$

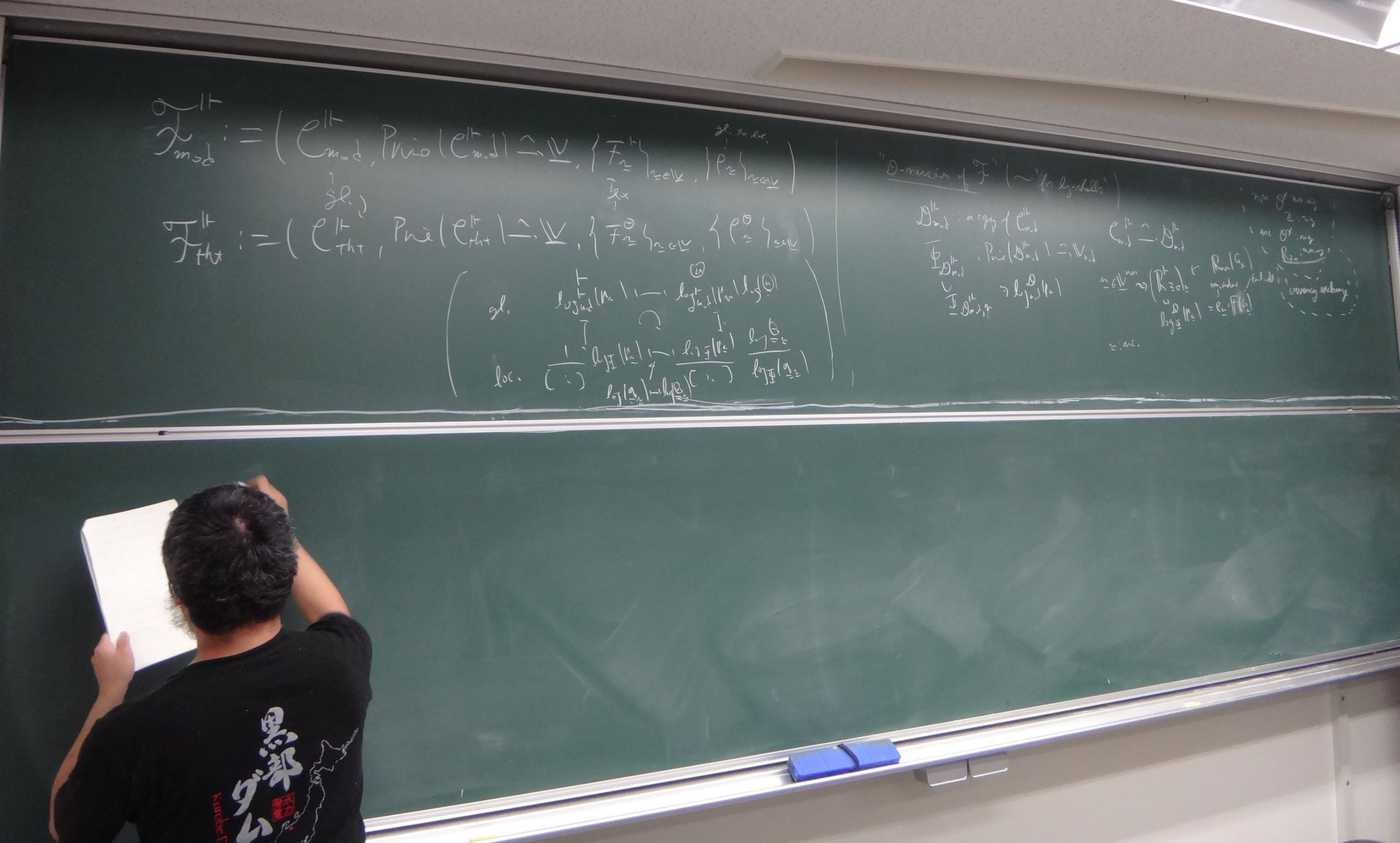
$$\rho^{\Theta}_{\underline{v}} : \Phi_{\mathcal{C}^{\Vdash}_{tht},v} \xrightarrow{\sim} \Phi^{rlf}_{\mathcal{C}^{\Theta}_{\underline{v}}}$$

$$\log^{\Vdash}_{mod}(p_v) \log(\underline{\underline{\Theta}}) \longmapsto \frac{\log_{\Phi}(p_{\underline{v}})}{[K_{\underline{v}} : (F_{mod})_v]} \left(\frac{\log(\underline{\underline{\Theta}})}{\log_{\Phi}(p_{\underline{v}})} \right) \leftarrow \text{non-arch}$$

$$\longmapsto \frac{1}{[K_{\underline{v}} : (F_{mod})_v]} \log_{\Phi}(p_{\underline{v}}) \log(\underline{\underline{\Theta}})$$

arch $p_v :=$ "e" $= 2.718281828459045 \ldots$

(for Θ-reg)



gl. to loc.

$$\mathfrak{F}^{\Vdash}_{\mathrm{mod}} := \left(\mathcal{C}^{\Vdash}_{\mathrm{mod}}, \mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{mod}}) \xrightarrow{\sim} \underline{\mathbb{V}}, \{\mathcal{F}^{\Vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, \{\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}\right)$$

gl. loc.

$$\mathfrak{F}^{\Vdash}_{\mathrm{tht}} := \left(\mathcal{C}^{\Vdash}_{\mathrm{tht}}, \mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{tht}}) \xrightarrow{\sim} \underline{\mathbb{V}}, \{\mathcal{F}^{\Theta}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, \{\rho^{\Theta}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}\right)$$

$$\left(\begin{array}{lll} \text{gl.} & \log^{\Vdash}_{\mathrm{mod}}(p_v) \mapsto & \log^{\Vdash}_{\mathrm{mod}}(p_v)/\log(\underline{\underline{\Theta}}) \\ & \quad\curvearrowright & \\ \text{loc.} & \frac{1}{[\;:\;]}\log_{\Phi}(p_v) \mapsto & \frac{\log_{\Phi}(p_v)}{[\;:\;]} \quad \frac{\log\underline{\underline{\Theta}}_v}{\log_{\Phi}(\underline{q}_v)} \\ & \log(\underline{q}_v) \mapsto \log(\underline{\underline{\Theta}}_v) & \end{array}\right)$$

"Θ-version of $\mathfrak{F}$" (~ "for log-shells")

$\mathcal{D}^{\Vdash}_{\mathrm{mod}}$: a copy of $\mathcal{C}^{\Vdash}_{\mathrm{mod}}$ $\quad \mathcal{C}^{\Vdash}_{\mathrm{mod}} \xrightarrow{\sim} \mathcal{D}^{\Vdash}_{\mathrm{mod}}$

$\Phi_{\mathcal{D}^{\Vdash}_{\mathrm{mod}}}$: $\mathrm{Prime}(\mathcal{D}^{\Vdash}_{\mathrm{mod}}) \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$

$\cup$

$\Phi_{\mathcal{D}^{\Vdash}_{\mathrm{mod},v}} \ni \log^{\mathcal{D}}_{\mathrm{mod}}(p_v)$

$\underline{v}\in\underline{\mathbb{V}}^{\mathrm{non}} \rightsquigarrow (\mathbb{R}^{\Vdash}_{\geq 0})_{\underline{v}} \leftarrow \mathbb{R}_{\mathrm{non}}(G_v)$ reg. indep. / Frob. dd.

$\cup$

$\log^{\mathcal{D}}_{\Phi}(p_v) = e_v^{-1} \cdot \boxed{F}(G_v)$

$\underline{v}$: arc.

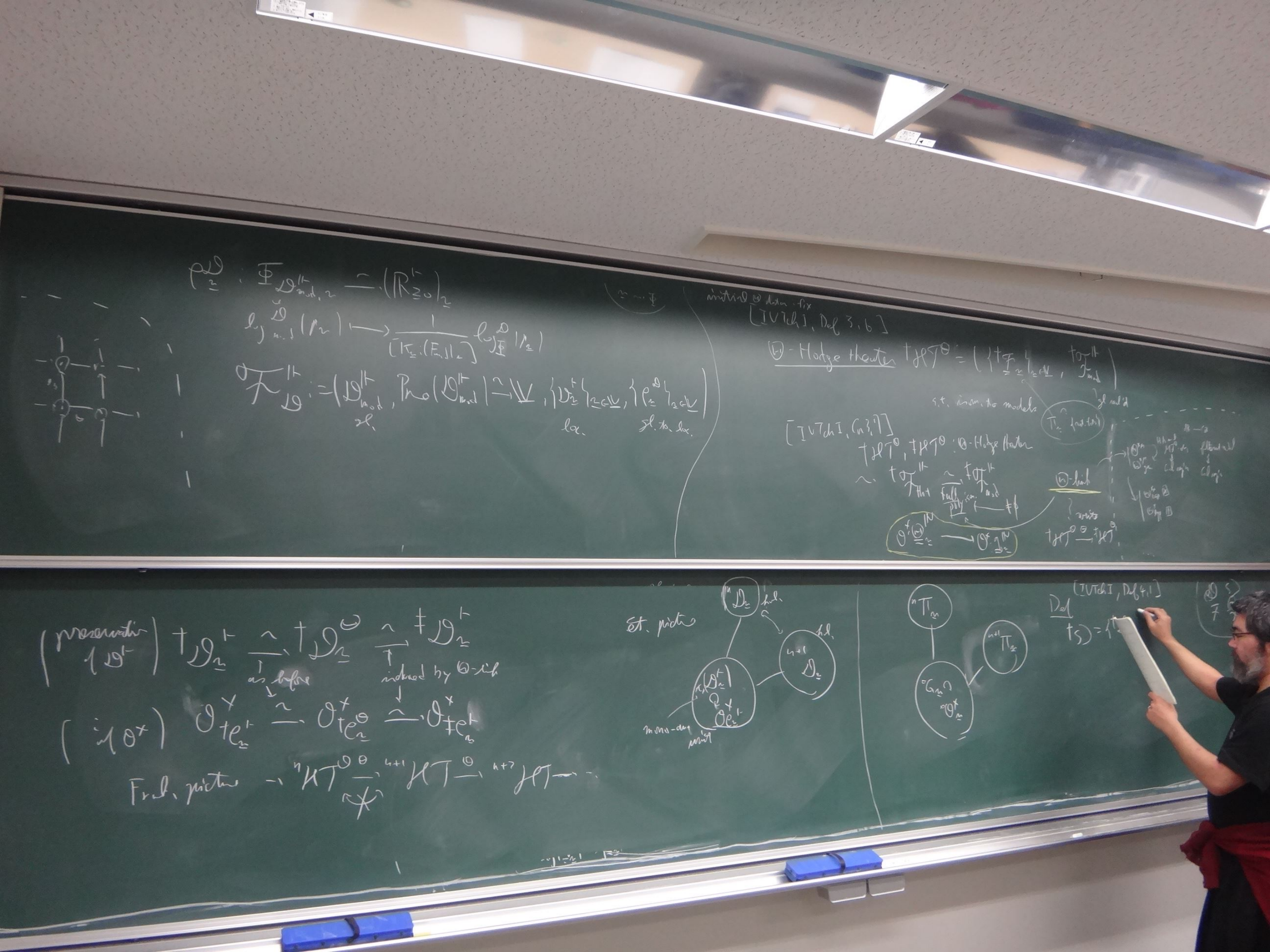
non $\mathcal{O}^{\times}$ arr. rig

$\mathbb{Z}$: rig

arc $\mathcal{O}^{\times}$: rig

$\mathbb{R}_{\geq 0}$: non-rig

"currency exchange"



$\rho^{\mathcal{D}}_{\underline{v}} : \Phi_{\mathcal{D}^{\vdash}_{\mathrm{mod},v}} \xrightarrow{\sim} (\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}}$

$\log^{\check{\mathcal{D}}}_{\mathrm{mod}}(p_{\underline{v}}) \longmapsto \frac{1}{[K_{\underline{v}} : (F_{\mathrm{mod}})_v]} \log^{\mathcal{D}}_{\Phi}(p_{\underline{v}})$

$^{\mathcal{D}}\mathcal{F}^{\Vdash}_{\mathcal{D}} := (\underset{\text{gl.}}{\mathcal{D}^{\vdash}_{\mathrm{mod}}}, \mathrm{Prime}(\mathcal{D}^{\vdash}_{\mathrm{mod}}) \xrightarrow{\sim} \underline{\mathbb{V}}, \underset{\text{loc.}}{\{\mathcal{D}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}}, \underset{\text{gl. to loc.}}{\{\rho^{\mathcal{D}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}})$

initial Θ data fix

[IUTchI, Def 3.6]

Θ-Hodge theater $^{\dagger}\mathcal{HT}^{\Theta} = (\{^{\dagger}\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, {}^{\dagger}\mathcal{F}^{\Vdash}_{\mathrm{mod}})$

s.t. isom. to models

gl. real'd

$\Pi_{\underline{v}}$ (rat. tors.)

[IUTchI, Cor 3.7]

$^{\dagger}\mathcal{HT}^{\Theta}, {}^{\ddagger}\mathcal{HT}^{\Theta}$: Θ-Hodge theater

$\rightsquigarrow {}^{\dagger}\mathcal{F}^{\Vdash}_{\mathrm{tht}} \xrightarrow[\text{full poly isom.}]{\sim} {}^{\ddagger}\mathcal{F}^{\Vdash}_{\mathrm{mod}} \longleftarrow \ddagger$

$\mathcal{O}^{\times}\cdot\Theta^{\mathbb{N}}_{\underline{v}} \longrightarrow \mathcal{O}^{\times}\cdot q^{\mathbb{N}}_{\underline{v}}$

Θ-link

write $^{\dagger}\mathcal{HT}^{\Theta} \xrightarrow{\Theta} {}^{\ddagger}\mathcal{HT}^{\Theta}$.

$\Theta^{\times\mu}$ Hodge $\mathcal{HT}^{\Theta}$ filtered

Θ^{gau}

$(\text{preservation of } \mathcal{D}^{\vdash})\quad {}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}} \xrightarrow{\sim} {}^{\dagger}\mathcal{D}^{\Theta}_{\underline{v}} \xrightarrow{\sim} {}^{\ddagger}\mathcal{D}^{\vdash}_{\underline{v}}$

as before

induced by Θ-link

$(\text{of } \mathcal{O}^{\times})\quad \mathcal{O}^{\times}_{^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}}} \xrightarrow{\sim} \mathcal{O}^{\times}_{^{\dagger}\mathcal{C}^{\Theta}_{\underline{v}}} \xrightarrow{\sim} \mathcal{O}^{\times}_{^{\ddagger}\mathcal{C}^{\vdash}_{\underline{v}}}$

Frob. picture $\to {}^{n}\mathcal{HT}^{\Theta} \xrightarrow{\Theta} {}^{n+1}\mathcal{HT} \xrightarrow{\Theta} {}^{n+2}\mathcal{HT} \longrightarrow \cdots$

Et. picture

[IUTchI, Def 4.1]

Def

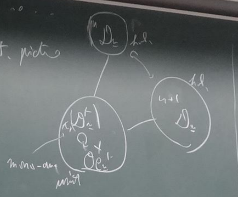
$^{\dagger}\mathfrak{S}) = \{^{\dagger}$

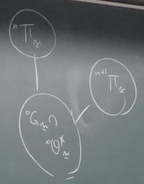
(preservation of $\mathcal{D}^{\vdash}$) ${}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}} \underset{\text{as before}}{\xrightarrow{\sim}} {}^{\dagger}\mathcal{D}^{\ominus}_{\underline{v}} \underset{\text{induced by } \Theta\text{-link}}{\xrightarrow{\sim}} {}^{\ddagger}\mathcal{D}^{\vdash}_{\underline{v}}$

(" of $\mathcal{O}^{\times}$) $\mathcal{O}^{\times}_{{}^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}}} \xrightarrow{\sim} \mathcal{O}^{\times}_{{}^{\dagger}\mathcal{C}^{\ominus}_{\underline{v}}} \xrightarrow{\sim} \mathcal{O}^{\times}_{{}^{\ddagger}\mathcal{C}^{\vdash}_{\underline{v}}}$

Frob. picture $\to {}^{n}\mathcal{HT}^{\mathcal{D}\text{-}\Theta} \underset{\cancel{\circlearrowleft}}{\xrightarrow{\Theta}} {}^{n+1}\mathcal{HT}^{\Theta} \xrightarrow{\Theta} {}^{n+2}\mathcal{HT} \longrightarrow \cdots$

et. picture





[IUTch I, Def 4.1]

Def

${}^{\dagger}\mathfrak{D} = \{{}^{\dagger}\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$

$\mathfrak{D} \subset$ $\mathcal{F}$ $\mathfrak{F}$

{ **$\mathcal{D}$-prime-strip** s.t. isom. to models

grp th'c

${}^{\dagger}\mathfrak{D} = \{{}^{\dagger}\underline{\mathcal{D}}_{\underline{v}}\} \longleftarrow$ "$\underline{\mathbb{C}}_{\underline{v}}$"

${}^{\dagger}\mathfrak{D}^{\vdash} = \{{}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$

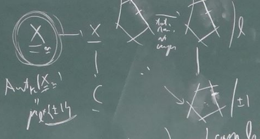
(moral of prime-strip $\underline{\mathbb{V}}$-prime = collection of ...)

$\mathcal{D}^{\vdash}$-prime-strip s.t. isom. to models

${}^{\dagger}\mathfrak{D} \underset{\text{grp th'c}}{\xrightarrow{\frown}} {}^{\dagger}\mathfrak{D}^{\vdash}$ mono-anabelian

${}^{\dagger}\mathcal{D}_{\underline{v}}$ $\underline{v}\in\underline{\mathbb{V}}^{\text{non}}$ $\underset{\text{grp th'c}}{\xrightarrow{\frown}}$ the set of cusps of ${}^{\dagger}\mathcal{D}_{\underline{v}}$

(anc. const)



label class of cusps of ${}^{\dagger}\mathcal{D}_{\underline{v}}$:= cusps of ${}^{\dagger}\mathcal{D}_{\underline{v}}$ lying / a non-zero cusp of ${}^{\dagger}\underline{\mathcal{D}}_{\underline{v}}$

$\underline{X}$ $\quad$ $\underline{C}$

$\mathrm{LabCusp}({}^{\dagger}\mathcal{D}_{\underline{v}}) := \{\text{cusp label classes}\}$

$\circlearrowleft\ \mathbb{F}_l^{*}$-torsor $\ni {}^{\dagger}\eta_{\underline{v}}$ corresp. $\underline{\epsilon}_{\underline{v}}$

$\mathfrak{D}^{\circledcirc} := \mathcal{B}(\underline{C}_{K})^{0} \underset{\text{grp th'c}}{\xrightarrow{\frown}} \overline{F}, \mathbb{V}(\overline{F})$

${}^{\dagger}\mathfrak{D}^{\circledcirc}$) isom.

${}^{\dagger}\mathfrak{D} = \{{}^{\dagger}\mathcal{D}_{\underline{v}}\}$ $\mathcal{D}$-prime-strip $\rightsquigarrow {}^{\dagger}\mathfrak{D} \xrightarrow{\text{(poly)}} {}^{\dagger}\mathfrak{D}^{\circledcirc}$

$\{{}^{\dagger}\mathcal{D}_{\underline{v}} \xrightarrow{\text{(poly)}} {}^{\dagger}\mathfrak{D}^{\circledcirc}\}_{\underline{v}\in\underline{\mathbb{V}}}$

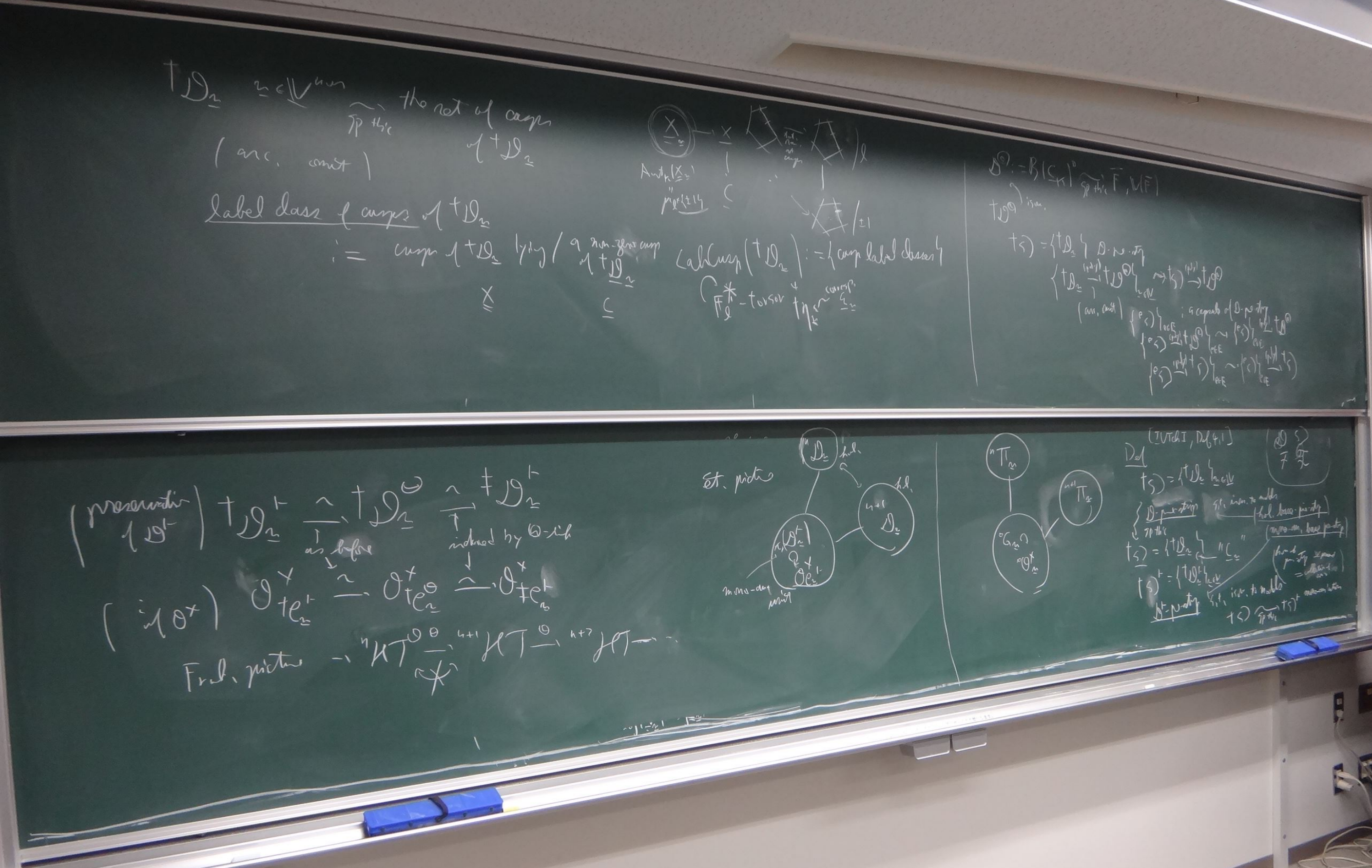
(anc. const)

; a capsule of $\mathcal{D}$-prime-strips ${}^{E}\mathfrak{D}$, ${}^{\dagger}\mathfrak{D}^{\circledcirc}$

$\{{}^{e}\mathfrak{D}\}_{e\in E}$, $\{{}^{e}\mathfrak{D} \xrightarrow{\text{(poly)}} {}^{\dagger}\mathfrak{D}^{\circledcirc}\}_{e\in E} \rightsquigarrow \{{}^{e}\mathfrak{D}\}_{e\in E}$

$\{{}^{e}\mathfrak{D} \xrightarrow{\text{(poly)}} {}^{\dagger}\mathfrak{D}\}_{e\in E} \rightsquigarrow \{{}^{e}\mathfrak{D}\}_{e\in E} \xrightarrow{\text{(poly)}} {}^{\dagger}\mathfrak{D}$

$\{{}^{e}\mathfrak{D}$ ...



$^{\dagger}\mathcal{D}_{\underline{v}}$ $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ $\underset{\text{grp-thic}}{\curvearrowright}$ the set of cusps of $^{\dagger}\mathcal{D}_{\underline{v}}$

(arc. const)

<u>label class of cusps</u> of $^{\dagger}\mathcal{D}_{\underline{v}}$

:= cusps of $^{\dagger}\mathcal{D}_{\underline{v}}$ lying / a non-zero cusp of $^{\dagger}\underline{\mathcal{D}}_{\underline{v}}$

$\underline{X}$ $\underline{C}$

$\underline{X}_{\underline{v}} \to \underline{X}$, $\underline{C}$

$\mathrm{Aut}_k(\underline{X}_{\underline{v}}) = \mu_{\ell} \times \{\pm 1\}$

/ℓ

/±1

LabCusp($^{\dagger}\mathcal{D}_{\underline{v}}$) := {cusp label classes}

$\curvearrowright \mathbb{F}_{\ell}^{*}$-torsor $\ni {}^{\dagger}\eta_{\underline{v}} \overset{\text{corresp.}}{\leftrightarrow} \underline{\xi}_{\underline{v}}$

$\mathcal{D}^{\circledcirc} := \mathcal{B}(\underline{C}_K)^{0} \underset{\text{grp-thic}}{\curvearrowright} \overline{F}, \mathcal{M}(\overline{F})$

$^{\dagger}\mathcal{D}^{\circledcirc}$ isom.

$^{\dagger}\mathfrak{S}) = \{^{\dagger}\mathcal{D}_{\underline{v}}\}$ D-prime-strip

$\{^{\dagger}\mathcal{D}_{\underline{v}} \xrightarrow{(\mathrm{poly})} {}^{\dagger}\mathcal{D}^{\circledcirc}\}_{\underline{v}\in\underline{\mathbb{V}}} \rightsquigarrow {}^{\dagger}\mathfrak{S}) \xrightarrow{(\mathrm{poly})} {}^{\dagger}\mathcal{D}^{\circledcirc}$

(arc, const)

$\{^{\epsilon}\mathfrak{S})\}_{\epsilon\in E}$; a capsule of D-prime-strip

$\{^{\epsilon}\mathfrak{S}) \xrightarrow{(\mathrm{poly})} {}^{\dagger}\mathcal{D}^{\circledcirc}\}_{\epsilon\in E} \rightsquigarrow \{^{\epsilon}\mathfrak{S})\}_{\epsilon\in E} \xrightarrow{(\mathrm{poly})} {}^{\dagger}\mathcal{D}^{\circledcirc}$

$\{^{\epsilon}\mathfrak{S}) \xrightarrow{(\mathrm{poly})} {}^{\dagger}\mathfrak{S})\}_{\epsilon\in E} \rightsquigarrow \{^{\epsilon}\mathfrak{S})\}_{\epsilon\in E} \xrightarrow{(\mathrm{poly})} {}^{\dagger}\mathfrak{S})$

(preservation of $\mathcal{D}^{\vdash}$) $^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}} \underset{\text{as before}}{\xrightarrow{\sim}} {}^{\dagger}\mathcal{D}^{\Theta}_{\underline{v}} \underset{\text{induced by } \Theta\text{-link}}{\xrightarrow{\sim}} {}^{\ddagger}\mathcal{D}^{\vdash}_{\underline{v}}$

(of $\mathcal{O}^{\times}$) $\mathcal{O}^{\times}_{^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}}} \xrightarrow{\sim} \mathcal{O}^{\times}_{^{\dagger}\mathcal{C}^{\Theta}_{\underline{v}}} \xrightarrow{\sim} \mathcal{O}^{\times}_{^{\ddagger}\mathcal{C}^{\vdash}_{\underline{v}}}$

Frob. picture $\longrightarrow {}^{n}\mathcal{HT}^{\Theta} \xrightarrow{\Theta} {}^{n+1}\mathcal{HT} \xrightarrow{\Theta} {}^{n+2}\mathcal{HT} \longrightarrow \cdots$

ét. picture

$^{n}\mathcal{D}_{\underline{v}}$ hol.

$^{n+1}\mathcal{D}_{\underline{v}}$ hol.

$\pi_1(\mathcal{D}^{\vdash}_{\underline{v}}) \curvearrowright \mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}$

mono-ana unit'g

$^{n}\Pi_{\underline{v}}$

$^{n+1}\Pi_{\underline{v}}$

$^{n}G_{\underline{v}} \curvearrowright {}^{n}\mathcal{O}^{\times}_{\underline{v}}$

[IUTchI, Def 4.1]

$\mathcal{D}$ $\mathfrak{S}$

$\mathcal{F}$ $\mathfrak{F}$

<u>Def</u>

$^{\dagger}\mathfrak{S}) = \{^{\dagger}\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$

<u>D-prime-strip</u> $\xleftarrow{\xi,\ \text{isom. to models}}$ (<u>hol. base-prime-strip</u>)

grp-thic

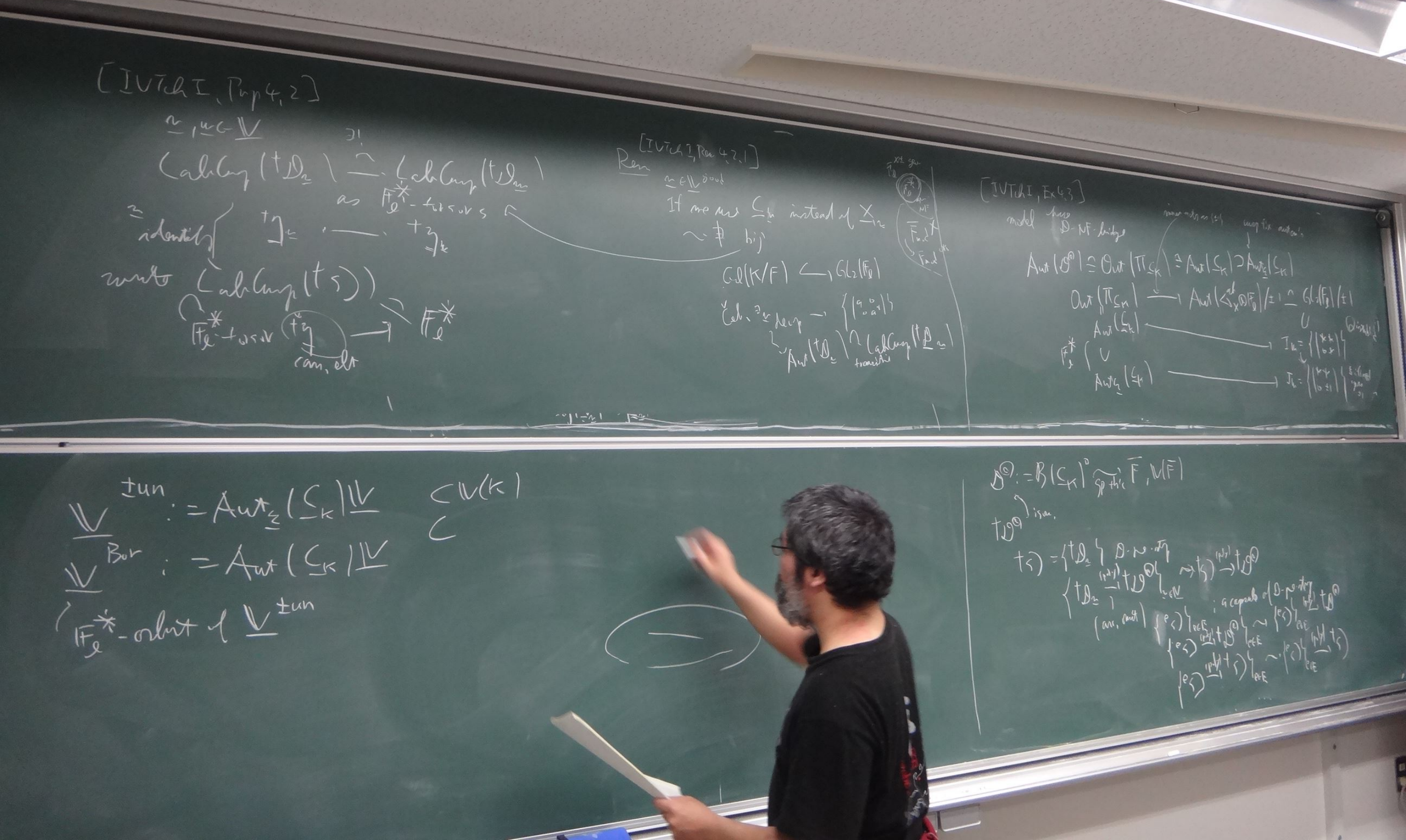
$^{\dagger}\mathfrak{S}\rangle = \{^{\dagger}\underline{\mathcal{D}}_{\underline{v}}\} \hookleftarrow$ "$\underline{C}_{\underline{v}}$" (<u>mono-an. base prime-strip</u>)

$^{\dagger}\mathfrak{S})^{\vdash} = \{^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$

<u>D⊢-prime-strip</u> ξ, isom. to models

(how to prime-strip = [illegible])

$^{\dagger}\mathfrak{S}) \underset{\text{grp-thic}}{\curvearrowright} {}^{\dagger}\mathfrak{S})^{\vdash}$ mono-an'ation



[IUTch I, Prop 4.2]

$\underline{v}, \underline{w} \in \underline{\mathbb{V}}$

$\mathrm{LabCusp}({}^\dagger\mathcal{D}_{\underline{v}}) \xrightarrow[\sim]{\exists!} \mathrm{LabCusp}({}^\dagger\mathcal{D}_{\underline{w}})$ as $\mathbb{F}_l^{*}$-torsors

identify: ${}^\dagger\eta_{\underline{v}} \longmapsto {}^\dagger\eta_{\underline{w}}$

onto $\mathrm{LabCusp}({}^\dagger\mathfrak{D}))$

$\mathbb{F}_l^{*}$-torsor $({}^\dagger\eta) \longrightarrow \mathbb{F}_l^{*}$

can. elt

Rem [IUTch I, Rem 4.2.1]

$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$

If we use $\underline{C}_K$ instead of $\underline{X}_{\underline{v}}$

$\rightsquigarrow$ ∄ bij

$\mathrm{Gal}(K/F) \hookrightarrow \mathrm{GL}_2(\mathbb{F}_l)$

Gal, $\exists\, \underline{v}$ decp $\rightarrow \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}\right\}$

$\hookrightarrow \mathrm{Aut}({}^\dagger\mathcal{D}_{\underline{v}}) \curvearrowright \mathrm{LabCusp}({}^\dagger\mathcal{D}_{\underline{v}})$ transitive

[IUTch I, Ex 4.3]

model here $\mathcal{D}$-NF-bridge

$\mathrm{Aut}(\mathcal{D}^{\circledcirc}) \cong \mathrm{Out}(\Pi_{\underline{C}_K}) \cong \mathrm{Aut}(\underline{C}_K) \supset \mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K)$

$\mathrm{Out}(\Pi_{\underline{C}_K}) \longrightarrow \mathrm{Aut}(\Delta_X^{ab} \otimes \mathbb{F}_l)/\pm1 \cong \mathrm{GL}_2(\mathbb{F}_l)/\pm1$

$\mathrm{Aut}(\underline{C}_K) \longrightarrow I_K = \left\{\begin{pmatrix} * & * \\ 0 & *\end{pmatrix}\right\}$

$\mathbb{F}_l^{*} \curvearrowright \cup$

$\mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K) \longrightarrow I_K = \left\{\begin{pmatrix} * & * \\ 0 & \pm1\end{pmatrix}\right\}$

$\underline{\mathbb{V}}^{\pm un} := \mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K) \cdot \underline{\mathbb{V}} \subset \mathbb{V}(K)$

$\underline{\mathbb{V}}^{Bor} := \mathrm{Aut}(\underline{C}_K) \cdot \underline{\mathbb{V}}$

$\mathbb{F}_l^{*}$-orbit of $\underline{\mathbb{V}}^{\pm un}$

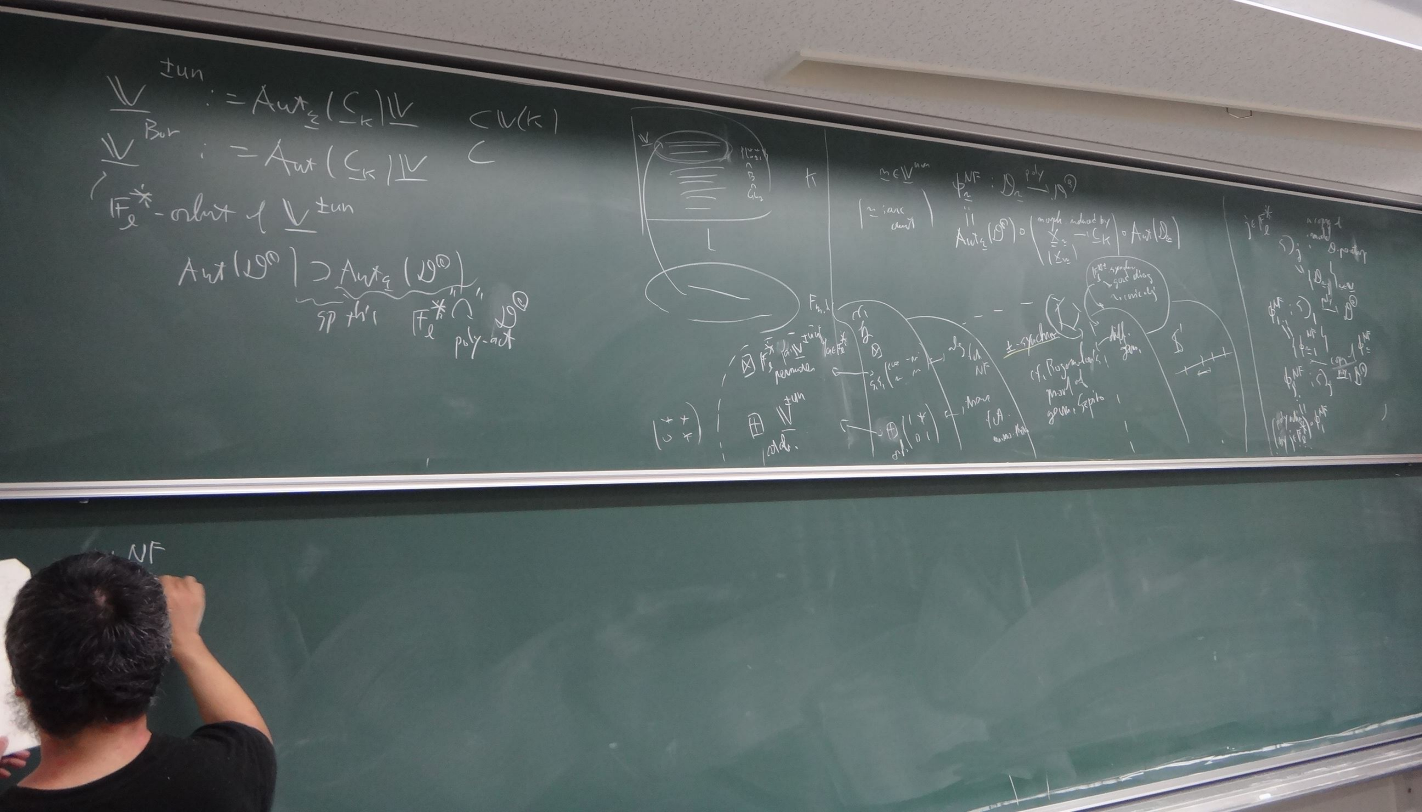
$\mathcal{D}^{\circledcirc} := \mathcal{B}(\underline{C}_K)^0$

${}^\dagger\mathcal{D}^{\circledcirc}$ isom.

${}^\dagger\mathfrak{D} = \{{}^\dagger\mathcal{D}_{\underline{v}}\}$ $\mathcal{D}$-prime-strip

$\{{}^\dagger\mathcal{D}_{\underline{v}} \xrightarrow{(pol)} {}^\dagger\mathcal{D}^{\circledcirc}\}_{\underline{v} \in \underline{\mathbb{V}}} \rightsquigarrow {}^\dagger\mathfrak{D} \xrightarrow{(pol)} {}^\dagger\mathcal{D}^{\circledcirc}$

(an. emb) ; a capsule of $\mathcal{D}$-prime-strips



$\underline{\mathbb{V}}^{\pm\mathrm{un}} := \mathrm{Aut}_{\epsilon}(\underline{C}_K) \cdot \underline{\mathbb{V}} \subset \mathbb{V}(K)$

$\underline{\mathbb{V}}^{\mathrm{Bor}} := \mathrm{Aut}(\underline{C}_K) \cdot \underline{\mathbb{V}} \subset$

$\mathbb{F}_l^{*}$-orbit of $\underline{\mathbb{V}}^{\pm\mathrm{un}}$

$\mathrm{Aut}(\mathcal{D}^{\circledcirc}) \supset \mathrm{Aut}_{\epsilon}(\mathcal{D}^{\circledcirc})$

sp th'i $\quad \mathbb{F}_l^{*} \curvearrowright \mathcal{D}^{\circledcirc}$

poly-act

$\underline{m} \in \underline{\mathbb{V}}^{\mathrm{non}}$

($\underline{m}$: arc chart)

$\phi^{\mathrm{NF}}_{\underline{m}} : \mathcal{D}_{\underline{m}} \xrightarrow{\mathrm{poly}} \mathcal{D}^{\circledcirc}$

$\parallel$

$\mathrm{Aut}_{\epsilon}(\mathcal{D}^{\circledcirc}) \circ \left(\begin{matrix} \text{morph. induced by} \\ \underline{X}_{\underline{m}} \to \underline{C}_K \end{matrix} \right) \circ \mathrm{Aut}(\mathcal{D}_{\underline{m}})$

$j \in \mathbb{F}_l^{*}$

n copy of model $\mathcal{D}$-prime strip

$\{\mathcal{D}_{\underline{m}}\}_{\underline{m} \in \underline{\mathbb{V}}}$

$\phi^{\mathrm{NF}}_1 : \mathcal{D}_1 \xrightarrow{\mathrm{poly}} \mathcal{D}^{\circledcirc}$

$\{\phi^{\mathrm{NF}}_{\underline{m}}\}$ $\xrightarrow{\mathrm{copy}}$ ϕ^{NF}_j

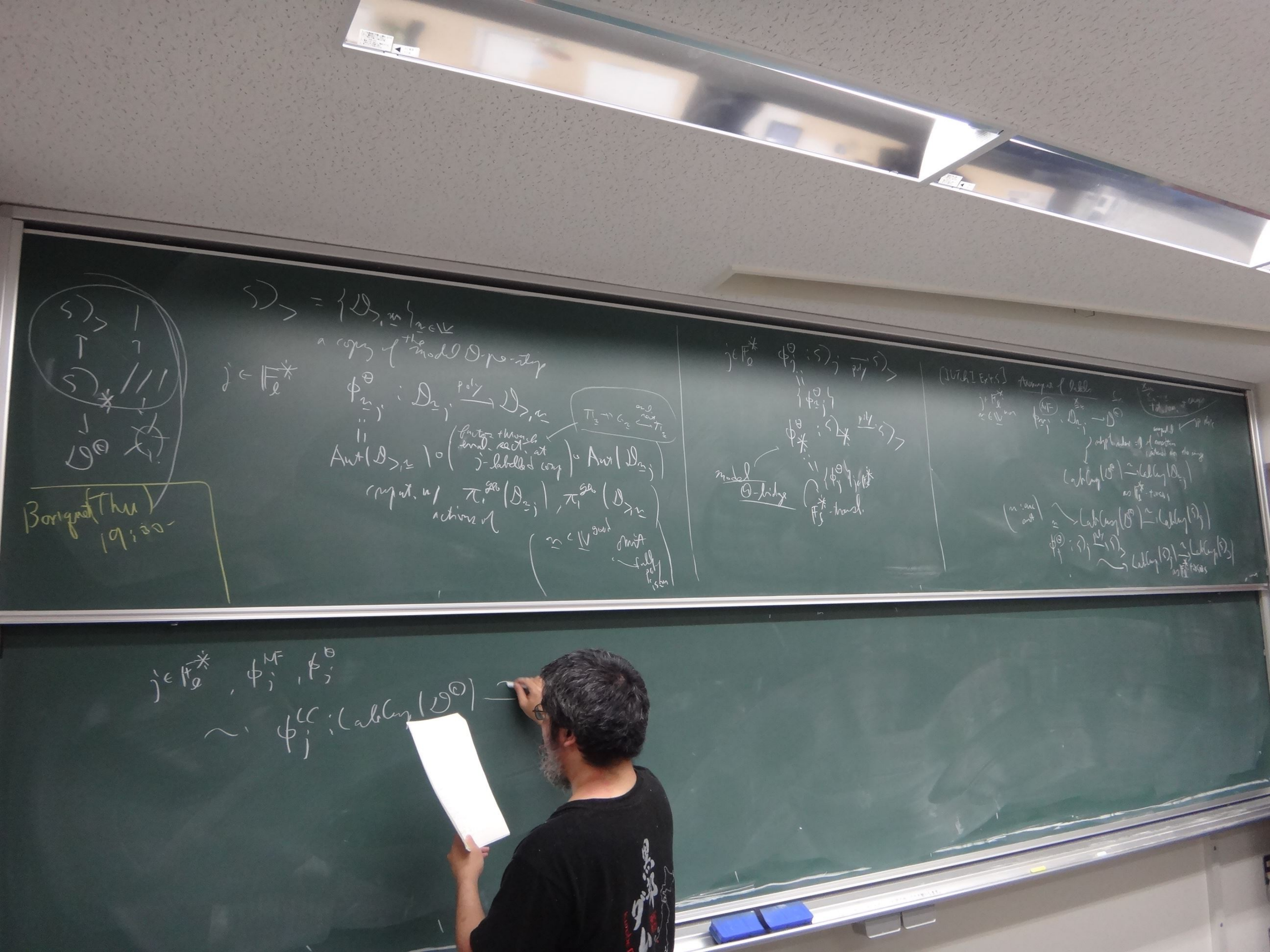
$\phi^{\mathrm{NF}}_j : \mathcal{D}_j \xrightarrow{\mathrm{poly}} \mathcal{D}^{\circledcirc}$

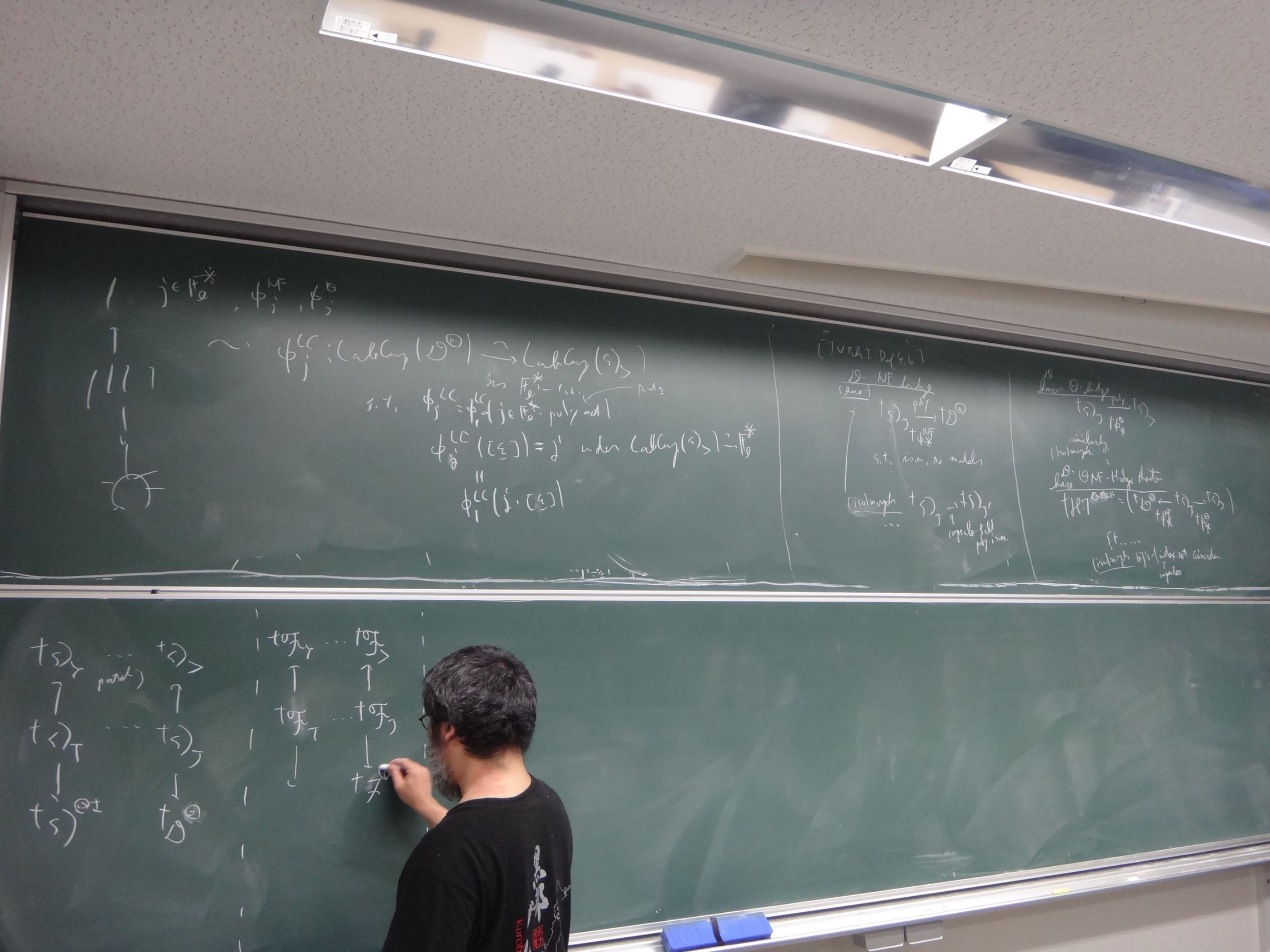
$\parallel$

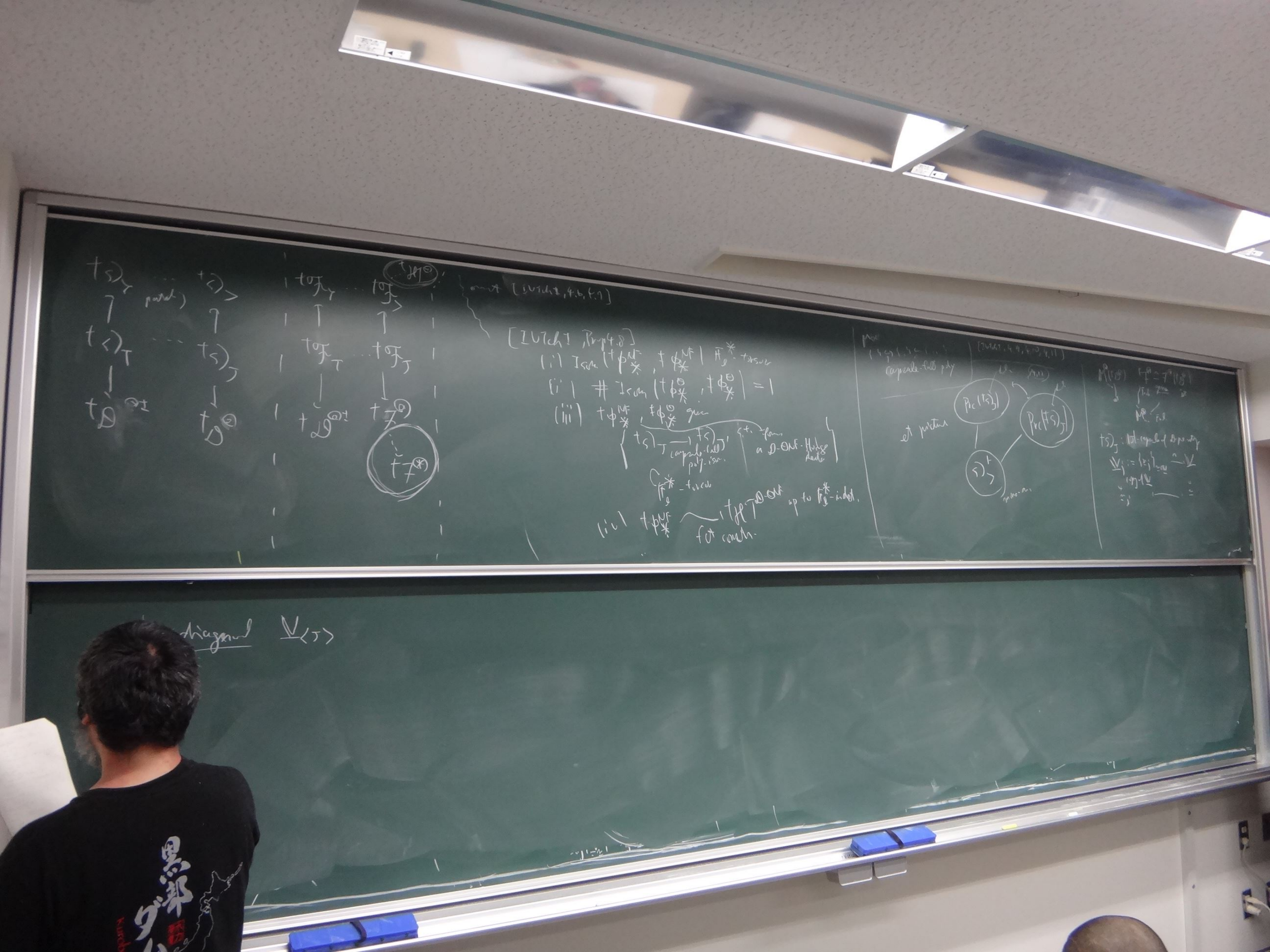
(poly autom. by $j \in \mathbb{F}_l^{*}$) $\circ\, \phi^{\mathrm{NF}}_1$

NF


model D-NF-bridge
[IUTchI Ex 4.4]
evaluation pt
for pt of order = 2
lying in 0 ∈ |F_l|-labelled comp.
|F_l| ∋ j -th comp
evaluation pt


Banquet (Thu) 19:30-
a copy of the model Θ-pilot
j ∈ F_l^*
poly
factors through
j-labelled comp
compat. w/ π_1^geo
actions of
model Θ-bridge
F_l^*-transl.
[IUTch I Ex 4.5]
Arrangement of labels
as F_l^*-torsors
LabCusp(D^⊖)
j ∈ F_l^*, φ_j^NF, φ_j^Θ
φ_j^LC : LabCusp(D^⊖)

$j \in \mathbb{F}_l^{*}$, ϕ_j^{NF}, ϕ_j^{Θ}
$\rightsquigarrow$ ϕ_j^{LC} : LabCusp$({}^{\dagger}\mathcal{D}^{\odot}) \xrightarrow{\sim}$ LabCusp$({}^{\dagger}\mathfrak{D}_{>})$
is $\mathbb{F}_l^{*}$-tors. Aut
s.t. $\phi_j^{LC} = \phi_1^{LC} \circ (j \in \mathbb{F}_l^{*}$ poly act$)$
$\phi_j^{LC}([\underline{\epsilon}]) = j$ under LabCusp$({}^{\dagger}\mathfrak{D}_{>}) \xrightarrow{\sim} \mathbb{F}_l^{*}$
$\phi_1^{LC}(j \cdot [\underline{\epsilon}])$
[IUTchI Def 4.6]
$\mathcal{D}$-NF bridge
(base-)
${}^{\dagger}\mathfrak{D}_J \xrightarrow{poly} {}^{\dagger}\mathcal{D}^{\odot}$
${}^{\dagger}\phi_*^{NF}$
s.t. isom. to models
(iso)morph ${}^{\dagger}\mathfrak{D}_J \to {}^{\dagger}\mathfrak{D}_{J'}$
capsule-full poly-isom.
base-Θ-bridge
${}^{\dagger}\mathfrak{D}_J \xrightarrow{poly} {}^{\dagger}\mathfrak{D}_{>}$
${}^{\dagger}\phi_*^{\Theta}$
similarly (iso)morph.
base-ΘNF-Hodge theater
${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta NF} = ({}^{\dagger}\mathcal{D}^{\odot} \leftarrow {}^{\dagger}\mathfrak{D}_J \to {}^{\dagger}\mathfrak{D}_{>})$
${}^{\dagger}\phi_*^{NF}$, ${}^{\dagger}\phi_*^{\Theta}$
s.t. ....
(iso)morph. bij's of index set coincide
capsules
${}^{\dagger}\mathfrak{D}_T$ ... ${}^{\dagger}\mathfrak{D}_{>}$
partial
${}^{\dagger}\mathfrak{D}_T$ ... ${}^{\dagger}\mathfrak{D}_J$
${}^{\dagger}\mathfrak{D}^{\circledast\pm}$
${}^{\dagger}\mathcal{D}^{\odot}$
${}^{\dagger}\mathcal{F}_T$ ... ${}^{\dagger}\mathcal{F}_{>}$
${}^{\dagger}\mathcal{F}_T$ ... ${}^{\dagger}\mathcal{F}_J$
${}^{\dagger}\mathcal{F}$



$\dagger\mathfrak{S})_{T}$ ... $\dagger\mathfrak{S})_{>}$

$\dagger\mathfrak{S})_{T}$ ... $\dagger\mathfrak{S})_{J}$

$\dagger\mathfrak{D}^{\circledcirc\pm}$ $\dagger\mathfrak{D}^{\circledcirc}$

$\dagger\mathcal{F}_{T}$ ... $\dagger\mathcal{F}_{>}$

$\dagger\mathcal{F}_{T}$ ... $\dagger\mathcal{F}_{J}$

$\dagger\mathcal{D}^{\circledcirc\pm}$ $\dagger\mathcal{F}^{\circledcirc}$

$\dagger\mathcal{F}^{\circledast}$

$\dagger\mathcal{HT}^{\circledcirc}$

omit [IUTch I, 4.6, 4.7]

[IUTch I, Prop 4.8]

(i) $\mathrm{Isom}(\dagger\phi^{NF}_{*}, \ddagger\phi^{NF}_{*})$ | $\mathbb{F}_l^{*}$-torsor

(ii) $\#\,\mathrm{Isom}(\dagger\phi^{\Theta}_{*}, \ddagger\phi^{\Theta}_{*}) = 1$

(iii) $\dagger\phi^{NF}_{*}$, $\ddagger\phi^{\Theta}_{*}$ given

$\{\dagger\mathfrak{S})_{T} \to \ddagger\mathfrak{S})_{J}$ capsule-full poly-isom. | s.t. forms a $\mathcal{D}$-ΘNF-Hodge theater$\}$

$\mathbb{F}_l^{*}$-torsor

(iv) $\dagger\phi^{NF}_{*} \mapsto \dagger\mathcal{HT}^{\mathcal{D}\text{-}\Theta NF}$ up to $\mathbb{F}_l^{*}$-indet. for const.

proof

capsule-full poly

[IUTch I, 4.9, 4.10, 4.11]

(4.11)

$\mathrm{Prc}(\dagger\mathfrak{S})_{J})$

$\mathrm{Prc}(\ddagger\mathfrak{S})_{J})$

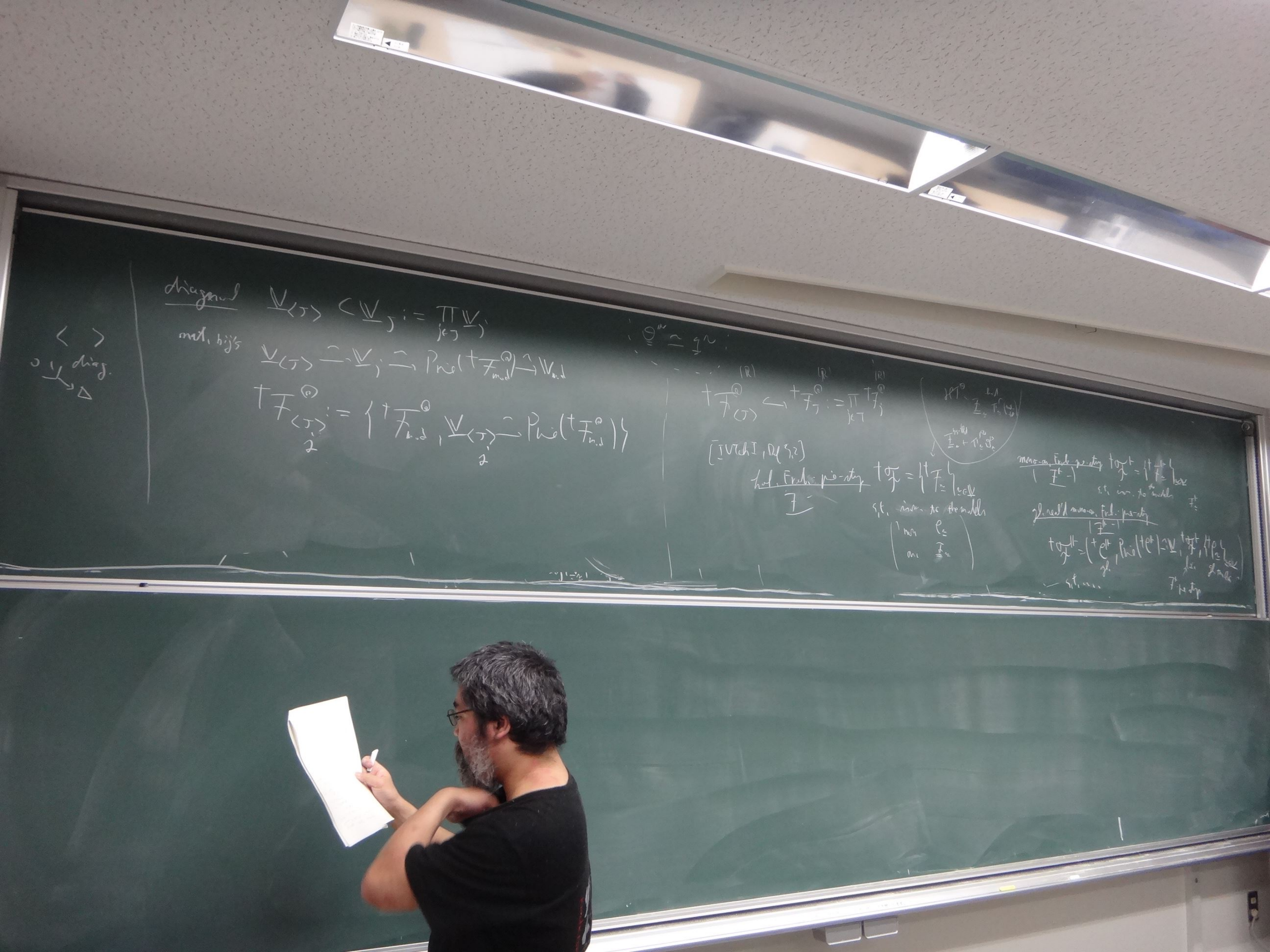
$\mathfrak{S})^{\pm}_{J}$

et picture

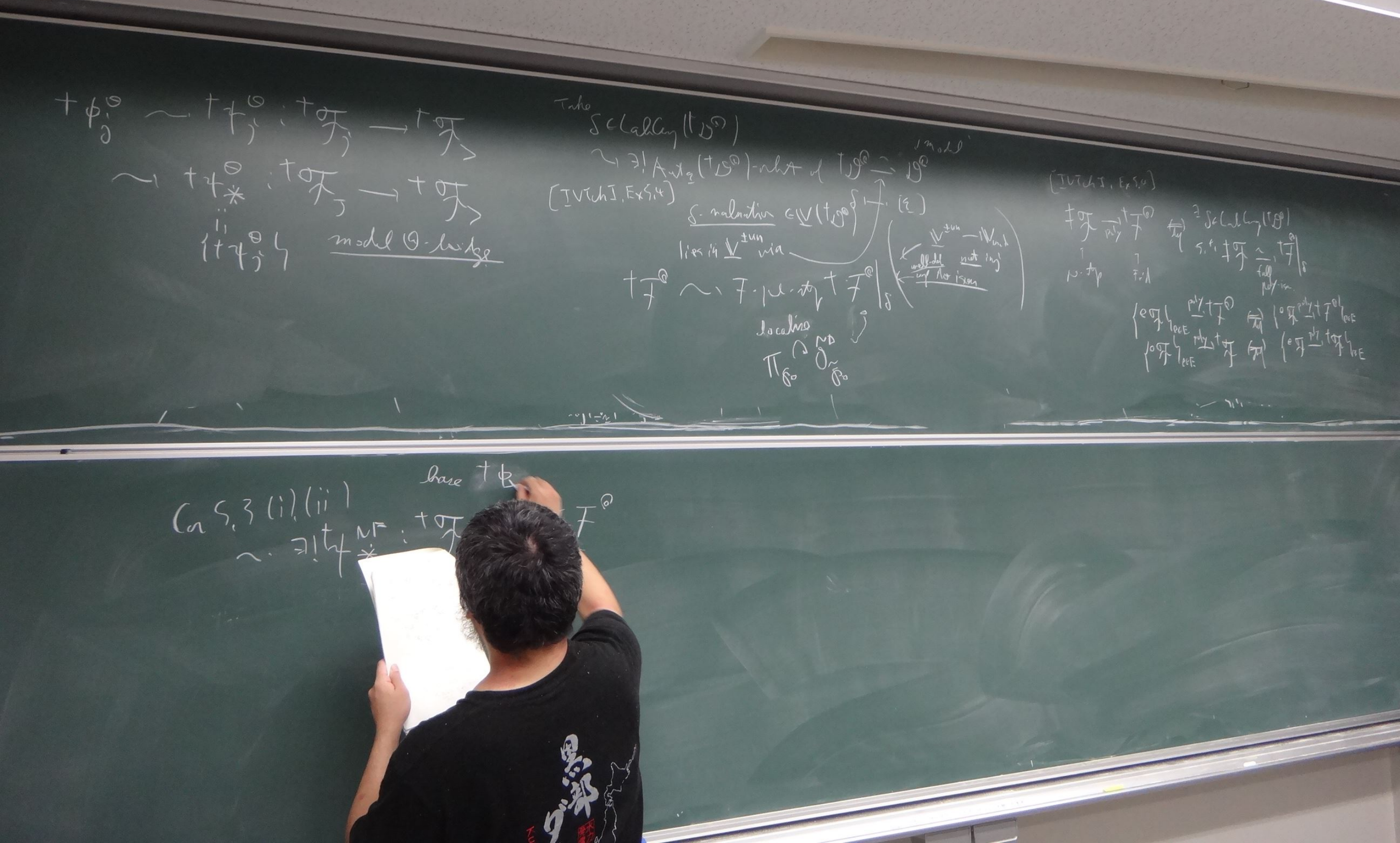
$\dagger\mathfrak{S})_{J}$: $|J|$-capsule of $\mathcal{D}$-prime-strip

$\mathbb{V}_{j} := \{v_{j}\}$

diagonal $\mathbb{V}_{\langle J \rangle}$


⟨ ⟩
0, 1, diag.
Δ
diagonal
$\underline{\mathbb{V}}_{\langle J\rangle} \subset \underline{\mathbb{V}}_J := \prod_{j\in J} \underline{\mathbb{V}}_j$
nat. bij's
$\underline{\mathbb{V}}_{\langle J\rangle} \xrightarrow{\sim} \underline{\mathbb{V}}_j \xrightarrow{\sim} \mathrm{Prime}({}^\dagger\mathcal{F}^{\circledR}_{\mathrm{mod}}) \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$
${}^\dagger\mathcal{F}^{\circledR}_{\langle J\rangle} := \{{}^\dagger\mathcal{F}^{\circledR}_{\mathrm{mod}}, \underline{\mathbb{V}}_{\langle J\rangle} \xrightarrow{\sim} \mathrm{Prime}({}^\dagger\mathcal{F}^{\circledR}_{\mathrm{mod}})\}$
${}^\dagger\mathcal{F}^{\circledR}_{\langle J\rangle} \hookrightarrow {}^\dagger\mathcal{F}^{\circledR}_J := \prod_{j\in J} {}^\dagger\mathcal{F}^{\circledR}_j$
[IUTch I, Def 3.2]
hol. Frob.-pre-strip
${}^\dagger\mathfrak{F} = \{{}^\dagger\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$
s.t. isom. to the models
mono-an. Frob.-pre-strip
s.t. isom. to the models
gl. real'd mono-an. Frob.-pre-strip
s.t. ...


[IUTch I, Cor 5.3]
(i) Isom
Base
(ii) Isom
(iii) Isom
(iv) Aut
need
diagonal
mult. bij's
[IUTch I, Def 5.2]
Frob.
s.t.
models
s.t. ....



$${}^{\dagger}\phi^{\Theta}_{j} \rightsquigarrow {}^{\dagger}\psi^{\Theta}_{j} : {}^{\dagger}\mathfrak{F}_{j} \to {}^{\dagger}\mathfrak{F}_{>}$$

$$\rightsquigarrow {}^{\dagger}\psi^{\Theta}_{*} : {}^{\dagger}\mathfrak{F}_{J} \to {}^{\dagger}\mathfrak{F}_{>}$$

$$\{{}^{\dagger}\psi^{\Theta}_{j}\}$$

<u>model Θ-bridge</u>

Take $\delta \in \mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\odot})$

$\rightsquigarrow \exists! \mathrm{Aut}_{\underline{\epsilon}}({}^{\dagger}\mathcal{D}^{\odot})$-orbit of ${}^{\dagger}\mathcal{D}^{\odot} \xrightarrow{\sim} \mathcal{D}^{\odot}$ (model)

[IUTchI, Ex5.4]

<u>δ-valuation</u> $\in \underline{\mathbb{V}}({}^{\dagger}\mathcal{D}^{\odot})^{\delta} \mapsto [\underline{\epsilon}]$

lies in $\underline{\mathbb{V}}^{\pm un}$ via

$\underline{\mathbb{V}}^{\pm un} \to \mathbb{V}_{mod}$

<u>well-def'd</u> <u>not inj</u>

<u>up to isom</u>

$${}^{\dagger}\mathcal{F}^{\odot} \rightsquigarrow \mathcal{F}\text{-prime-strip } {}^{\dagger}\mathcal{F}^{\odot}|_{\delta}$$

localize

$$\Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\triangleright}_{\underline{\tilde{v}}}$$

[IUTchI, Ex5.4]

$${}^{\ddagger}\mathfrak{F} \xrightarrow[\text{poly}]{} {}^{\dagger}\mathcal{F}^{\odot} \overset{\text{def}}{\iff} \exists \delta \in \mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\odot}) \text{ s.t. } {}^{\ddagger}\mathfrak{F} \underset{\substack{\text{full}\\ \text{poly-isom}}}{\xrightarrow{\sim}} {}^{\dagger}\mathcal{F}^{\odot}|_{\delta}$$

$\mathcal{F}$-prime-strip $\mathcal{F}^{\odot}$

$$\{{}^{e}\mathfrak{F}\}_{e\in E} \xrightarrow{\text{poly}} {}^{\dagger}\mathcal{F}^{\odot} \overset{\text{def}}{\iff} \{{}^{e}\mathfrak{F} \xrightarrow{\text{poly}} {}^{\dagger}\mathcal{F}^{\odot}\}_{e\in E}$$

$$\{{}^{e}\mathfrak{F}\}_{e\in E} \xrightarrow{\text{poly}} {}^{\dagger}\mathfrak{F} \overset{\text{def}}{\iff} \{{}^{e}\mathfrak{F} \xrightarrow{\text{poly}} {}^{\dagger}\mathfrak{F}\}_{e\in E}$$

Cor 5.3 (i), (ii)

have ${}^{\dagger}\Psi$

$$\rightsquigarrow \exists! {}^{\dagger}\psi^{NF}_{*} : {}^{\dagger}\mathfrak{F} \quad \mathcal{F}^{\odot}$$

Cor 5.3 (i), (ii) ${}^\dagger\phi^{NF}_{*}$ given

$\rightsquigarrow \exists!\, {}^\dagger\psi^{NF}_{*} : {}^\dagger\mathfrak{F}_J \longrightarrow {}^\dagger\mathcal{F}^{\circledcirc}$ $/ {}^\dagger\psi^{NF}_{*}$

$$\begin{tikzcd}
{}^\dagger\mathfrak{F}_{\langle J\rangle} \arrow[r, "\text{choose}"] & {}^\dagger\mathcal{F}^{\circledcirc} \arrow[r, "\sim", "\text{induce}"'] & {}^\dagger\mathcal{F}^{\circledcirc}_{\text{mod}} \arrow[r, "\sim"] & {}^\dagger\mathcal{F}^{\circledcirc}_{\langle J\rangle} \arrow[r, "\text{restr.}"] & {}^\dagger\mathcal{F}_{\underline{v}\langle J\rangle} \\
& & \text{gl.} & & \text{inverse of } {}^\dagger\mathfrak{F}_{\langle J\rangle} \xrightarrow{\sim} {}^\dagger\mathcal{F}^{\circledcirc}|_{\underline{v}} \arrow[u] \quad \text{loc.}
\end{tikzcd}$$

indep.

$\rightsquigarrow {}^\dagger\mathcal{F}^{\circledcirc}_{\text{mod}} \xrightarrow{\sim} {}^\dagger\mathcal{F}^{\circledcirc}_{\langle J\rangle} \longrightarrow {}^\dagger\mathfrak{F}_{\langle J\rangle}$

have ${}^\dagger\mathcal{D}^{\circledcirc} \to {}^\dagger\mathcal{D}^{\circledast}$

[IUTch I Def 5.5]

NF-bridge $\left({}^\dagger\mathfrak{F}_J \xrightarrow[{}^\dagger\psi^{NF}_{*}]{} {}^\dagger\mathcal{F}^{\circledcirc} \dashrightarrow {}^\dagger\mathcal{F}^{\circledast}\right)$

s.t. ....

Θ-bridge $\left({}^\dagger\mathfrak{F}_J \xrightarrow[{}^\dagger\psi^{\Theta}_{*}]{} {}^\dagger\mathfrak{F}_{>} \dashrightarrow {}^\dagger\mathcal{HT}^{\Theta}\right)$ $\left({}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}} \text{ in the def. } {}^\dagger\underline{\underline{\mathcal{F}}}^{\text{n-Ell}}_{\underline{v}} \ \vdots \right)$

s.t. ....

tw. F-prestrip

[isomorph ....

[IUTchI, Def 5.5]

ΘNF-Hodge theater

$${}^{\dagger}\mathcal{HT}^{\Theta NF} = \left({}^{\dagger}\mathcal{F}^{\circledast} \leftarrow \cdots {}^{\dagger}\mathcal{F}^{\circledcirc} \xleftarrow{{}^{\dagger}\psi^{NF}_{*}} {}^{\dagger}\mathfrak{D}_{J} \xrightarrow{{}^{\dagger}\psi^{\Theta}_{*}} {}^{\dagger}\mathfrak{D}_{>} \to \cdots \to {}^{\dagger}\mathcal{HT}^{\Theta}\right)$$

s.t. . . .

[IUTchI, Cor 5.6]

(isolm pt . . . .

i) $\mathrm{Isom}({}^{\dagger}\mathcal{HT}^{\Theta}, {}^{\ddagger}\mathcal{HT}^{\Theta}) \xrightarrow{\sim} \mathrm{Isom}({}^{\dagger}\mathfrak{D}_{>}, {}^{\ddagger}\mathfrak{D}_{>})$

ii) $\mathrm{Isom}({}^{\dagger}\psi^{NF}_{*}, {}^{\ddagger}\psi^{NF}_{*}) \xrightarrow{\sim} \mathrm{Isom}({}^{\dagger}\phi^{NF}_{*}, {}^{\ddagger}\phi^{NF}_{*})$

$\quad {}^{\dagger}\psi^{\Theta}_{*} \quad {}^{\ddagger}\psi^{\Theta}_{*} \qquad\qquad {}^{\dagger}\phi^{\Theta}_{*} \quad {}^{\ddagger}\phi^{\Theta}_{*}$

$\mathrm{Isom}({}^{\dagger}\mathcal{HT}^{\Theta NF}, {}^{\ddagger}\mathcal{HT}^{\Theta NF}) \xrightarrow{\sim} \mathrm{Isom}({}^{\dagger}\mathcal{HT}^{D\text{-}\Theta NF}, {}^{\ddagger}\mathcal{HT}^{D\text{-}\Theta NF}$

iii) ${}^{\dagger}\psi^{NF}_{*}, {}^{\ddagger}\psi^{\Theta}_{*}$ given

$\{{}^{\dagger}\mathfrak{D}_{J} \underset{\text{cap-full poly-isom.}}{\longrightarrow} {}^{\ddagger}\mathfrak{D}_{J},$ s.t. form ΘNF-HT$\}$

$\circlearrowleft \mathbb{F}_l^{*}$-torsor.

$$\left(\begin{array}{l} \mathfrak{D}^{\circledcirc} = \mathcal{B}(\underline{C}_K)^{0} \\ \mathfrak{D}^{\circledcirc\pm} = \mathcal{B}(\underline{X}_K)^{0} \end{array}\right.$$

$\boxed{+}$

[IUTchI, Def 6.1]

$\mathbb{F}_l^{\rtimes\pm} \twoheadrightarrow \{\pm1\}$

pos $\mapsto$ +1

neg. $\mapsto$ −1

E : $\underline{\mathbb{F}_l^{\pm}\text{-gp}}$ $\overset{\text{def}}{\Longleftrightarrow}$ E : a set w/ $\{\pm1\}$-orbit of bij's $E \xrightarrow{\sim} \mathbb{F}_l$

($\leftarrow$ an equiv. $\mathbb{F}_l$-module str.)

$\circlearrowleft \mathbb{F}_l^{\rtimes\pm}$

T : $\underline{\mathbb{F}_l^{\pm}\text{-torsor}}$ $\overset{\text{def}}{\Longleftrightarrow}$ T : a set w/ an $\mathbb{F}_l^{\rtimes\pm}$-orbit of bij's $T \xrightarrow{\sim} \mathbb{F}_l$

$\rightsquigarrow \mathrm{Aut}_{+}(T) \subset \mathrm{Aut}_{\pm}(T)$ index 2

$\mathrm{Aut}_{+}(\mathbb{F}_l) := \{z \mapsto z+\lambda\}$

$\wedge$

$\mathrm{Aut}_{\pm}(\mathbb{F}_l) := \{z \mapsto \pm z+\lambda\}$

pos. autom.

$\underline{\pm\text{-label class of cusps of}}$ ${}^{\dagger}\mathfrak{D}_{\pm}$

$\underline{X}_{\pm}$

$\underline{X}_{,\pm}$

$\underline{X}_{\pm}$

$\Leftrightarrow$ a set of cusps lying over a cusp

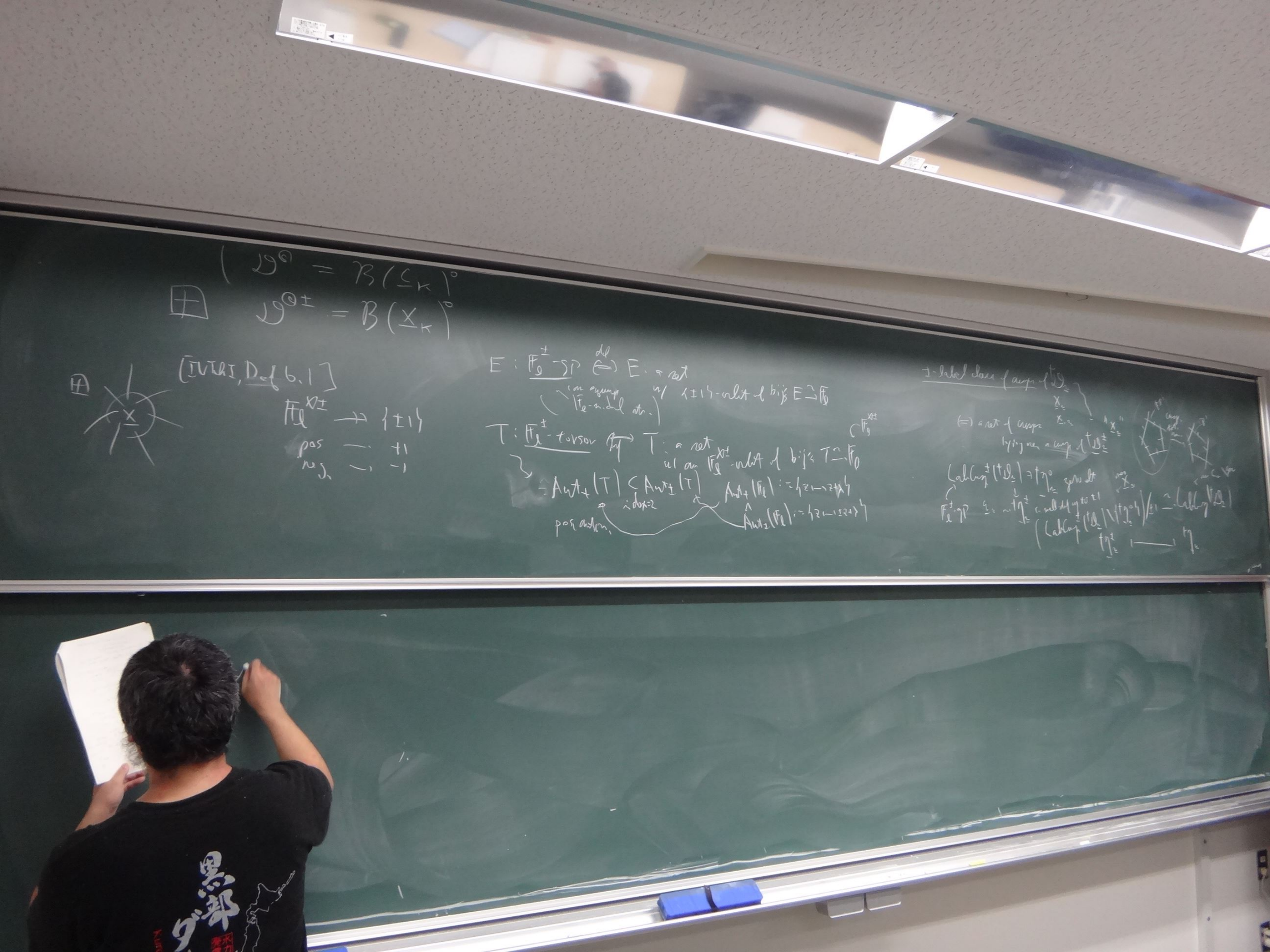
${}^{\dagger}\eta^{\pm}$

$\mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\pm}) \ni {}^{\dagger}\eta^{0}_{\pm}$ generator

$\updownarrow \quad {}^{\dagger}\eta^{\pm}_{\pm}$ ($\leftarrow$ well-def up to ±1)

$\mathbb{F}_l^{\pm}$-gp $\quad \xi_{\pm} \sim$

$(\mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\pm}) \setminus \{{}^{\dagger}\eta^{\pm}_{\pm}$



($\mathfrak{D}^{\circledcirc} = \mathcal{B}(\underline{C}_K)^0$

⊞ $\mathfrak{D}^{\circledcirc\pm} = \mathcal{B}(\underline{X}_K)^0$

[IUTchI, Def 6.1]

⊞

$\mathbb{F}_l^{\rtimes\pm} \twoheadrightarrow \{\pm 1\}$

pos ↦ +1

neg. ↦ −1

E : $\underline{\mathbb{F}_l^{\pm}\text{-gp}} \overset{\text{def}}{\Leftrightarrow}$ E : a set w/ $\{\pm1\}$-orbit of bij's $E \xrightarrow{\sim} \mathbb{F}_l$

(an equiv. $\mathbb{F}_l$-module str.)

T : $\underline{\mathbb{F}_l^{\pm}\text{-torsor}} \overset{\text{def}}{\Leftrightarrow}$ T : a set w/ an $\mathbb{F}_l^{\rtimes\pm}$-orbit of bij's $T \xrightarrow{\sim} \mathbb{F}_l$ $\circlearrowleft \mathbb{F}_l^{\rtimes\pm}$

$\rightsquigarrow \mathrm{Aut}_+(T) \subset \mathrm{Aut}_\pm(T)$ index=2

pos. autom.

$\mathrm{Aut}_+(\mathbb{F}_l) := \{z \mapsto z+\lambda\}$

$\widehat{\mathrm{Aut}}_\pm(\mathbb{F}_l) := \{z \mapsto \pm z+\lambda\}$

$\underline{\pm\text{-label class of cusps of } {}^\dagger\mathcal{D}_{\underline{v}}}$ $\underline{X}_{\underline{v}}$ $\underline{X}_{-\underline{v}}$ "$\underline{X}_{\underline{v}}$"

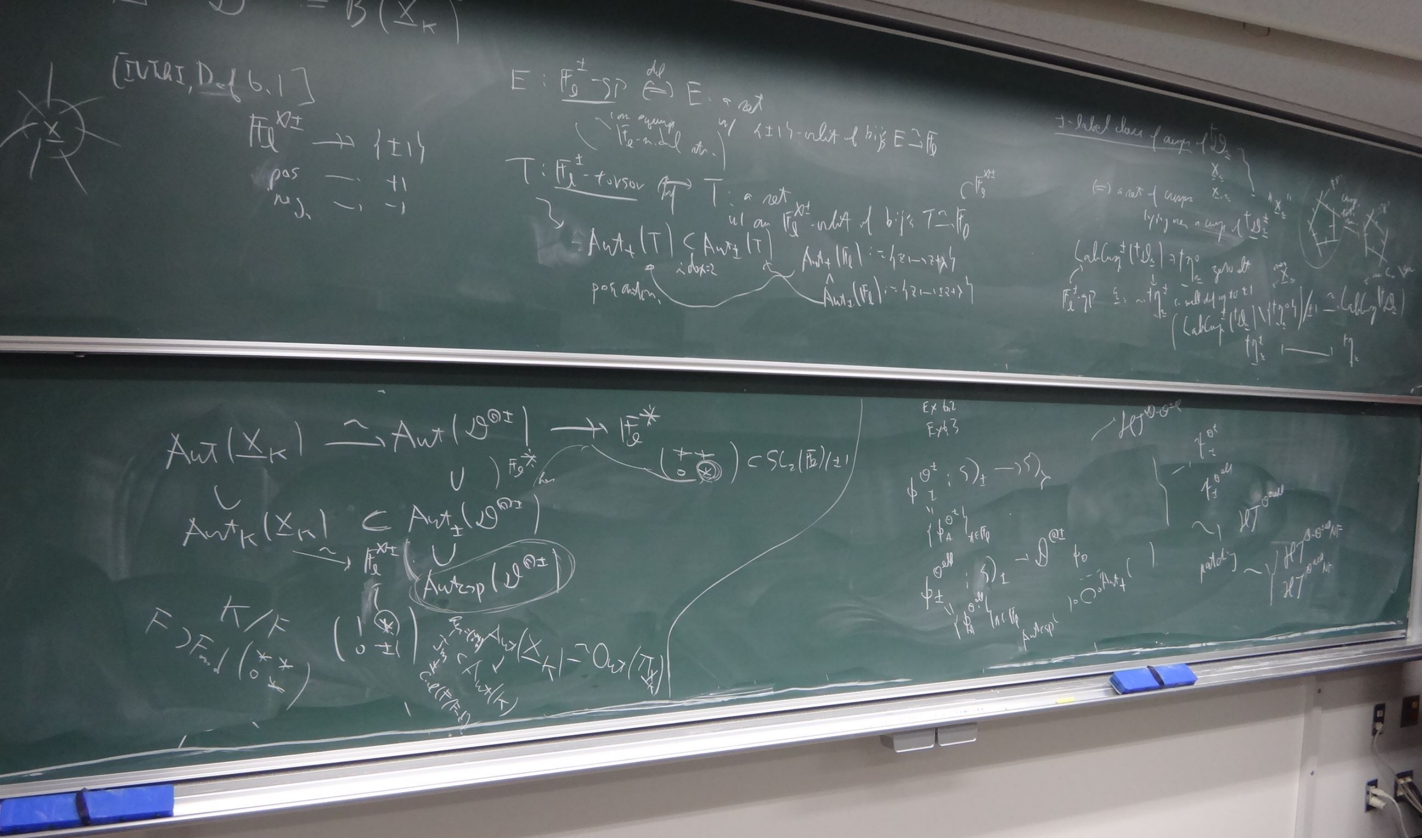
⟺ a set of cusps lying over a cusp of ${}^\dagger\mathcal{D}^{\pm}_{\underline{v}}$

$\mathrm{LabCusp}^\pm({}^\dagger\mathcal{D}_{\underline{v}}) \ni {}^\dagger\eta^0_{\underline{v}}$ zero elt

$\mathbb{F}_l^\pm$-gp $\underline{t}_{\underline{v}} \mapsto {}^\dagger\eta^\pm_{\underline{v}}$: well-def up to ±1

$(\mathrm{LabCusp}^\pm({}^\dagger\mathcal{D}_{\underline{v}}) \setminus \{{}^\dagger\eta^0_{\underline{v}}\})/\{\pm1\} \xrightarrow{\sim} \mathrm{LabCusp}({}^\dagger\mathcal{D}_{\underline{v}})$

${}^\dagger\eta^\pm_{\underline{v}} \longmapsto {}^\dagger\eta_{\underline{v}}$



$\mathcal{D} = \mathcal{B}(\underline{X}_K)$

[IUTchI, Def 6.1]

$\mathbb{F}_l^{\rtimes\pm} \twoheadrightarrow \{\pm 1\}$

pos $\mapsto +1$

neg. $\mapsto -1$

$E : \underline{\mathbb{F}_l^{\pm}\text{-grp}} \overset{\text{def}}{\Longleftrightarrow} E$: a set w/ $\{\pm 1\}$-orbit of bij's $E \xrightarrow{\sim} \mathbb{F}_l$

($\leftarrow$ an equivp $\mathbb{F}_l$-mod'l str.)

$T : \underline{\mathbb{F}_l^{\pm}\text{-torsor}} \overset{\text{def}}{\Longleftrightarrow} T$: a set w/ an $\mathbb{F}_l^{\rtimes\pm}$-orbit of bij's $T \xrightarrow{\sim} \mathbb{F}_l \curvearrowleft \mathbb{F}_l^{\rtimes\pm}$

$\rightsquigarrow \mathrm{Aut}_+(T) \subset \mathrm{Aut}_\pm(T)$ index 2

pos autom.

$\mathrm{Aut}_+(\mathbb{F}_l) := \{z \mapsto z + \lambda\}$

$\hat{\mathrm{Aut}}_\pm(\mathbb{F}_l) := \{z \mapsto \pm z + \lambda\}$

$\pm$-label class of cusps of $^\dagger\mathcal{D}_\pm$

$\underline{X}_\pm$

$(=)$ a set of cusps $\underline{X}_{\pm}$ "$\underline{X}_\pm$"

lying over a cusp of $^\dagger\mathcal{D}_\pm^{\pm}$

$\mathrm{LabCusp}^{\pm}(^\dagger\mathcal{D}_\pm) \ni {}^\dagger\eta^0_\pm$ zero elt

$\mathbb{F}_l^{\pm}\text{-grp} \quad \xi_\pm \rightsquigarrow {}^\dagger\eta^{\pm}_\pm$ (well def up to ± 1)

$(\mathrm{LabCusp}^{\pm}(^\dagger\mathcal{D}_\pm) \setminus \{{}^\dagger\eta^0_\pm\})/\{\pm 1\} \xrightarrow{\sim} \mathrm{LabCusp}(^\dagger\mathcal{D}_\pm)$

${}^\dagger\eta^{\pm}_\pm \longmapsto {}^\dagger\eta_\pm$

cusp, lab'l, ... on C_K $\underline{X}_\pm$

$\mathrm{Aut}(\underline{X}_K) \xrightarrow{\sim} \mathrm{Aut}(\mathcal{D}^{\circledcirc\pm}) \twoheadrightarrow \mathbb{F}_l^{*}$

$\cup \qquad\qquad \cup \;) \; \mathbb{F}_l^{*}$

$\mathrm{Aut}_K(\underline{X}_K) \subset \mathrm{Aut}_\pm(\mathcal{D}^{\circledcirc\pm})$

$\xrightarrow{\sim} \mathbb{F}_l^{\rtimes\pm} \; (\; \cup$

$\boxed{\mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm})}$

K/F

$F \supset F_{\mathrm{mod}}$ $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

$\begin{pmatrix} 1 & \circledast \\ 0 & \pm 1 \end{pmatrix}$

$\begin{pmatrix} \pm & \pm \\ 0 & \circledast \end{pmatrix} \subset SL_2(\mathbb{F}_l)/\{\pm 1\}$

$\mathrm{Aut}(\underline{X}_K) \xrightarrow{\sim} \mathrm{Out}(\Pi_{\underline{X}_K})$

$\cup$

$\mathrm{Aut}(K)$

$\mathrm{Gal}(F/F_{\mathrm{mod}})$

Ex 6.2

Ex 6.3

$\phi^{\Theta^\pm}_{\pm} ; \mathfrak{I})_\pm \to \mathfrak{I})_\succ$

$\{\phi^{\Theta^\pm}_A\}_{A \in \mathbb{F}_l}$

$\phi^{\Theta^{\mathrm{ell}}}_{\pm} ; \mathfrak{I})_\pm \to \mathcal{D}^{\circledcirc\pm}$

$\{\phi^{\Theta^{\mathrm{ell}}}_A\}_{A \in \mathbb{F}_l}$

ϕ_0

$1 \circ \bigcirc \circ \mathrm{Aut}_+($ $)$

$\mathrm{Aut}_{\mathrm{csp}}($

$\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$

$\mathcal{T}^{\Theta^\pm}_\pm$

$\mathcal{T}^{\Theta^{\mathrm{ell}}}_\pm$

$\rightsquigarrow \mathcal{HT}^{\Theta^{\pm\mathrm{ell}}}$

$\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$

gluing $\rightsquigarrow$ $\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$